Topological recursion in matrix models and problems of algebraic geometry

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Matrix models are, on one hand, easily accessible for investigation being just finite-dimensional integrals over (commonly Hermitian) matrices and, on the other hand, manifest extremely rich mathematical structure being subject simultaneously to nonlinear equations of hierarchies of integrable structures and to linear differential equations originated from conformal symmetries. During last 30 years of development of the matrix model theory, these models had found a variety of applications: from geometrical structures on moduli spaces of Riemann surfaces to recent papers (Alday–Gaiotto–Tachikawa hypothesis) relating generalisations of matrix models to conformal blocks of the quantum Liouville theory. We derive and exploit the method of topological recursion [4] in applications to matrix models and models of cohomological field theories. This course will be therefore a goof introduction to this actively developing branch of knowledge.

3rd and 4th quarters of 2014-15; 2 hours of lectures per week.

The program of the course:

1. Integrals over $N \times N$-matrices and the asymptotic $1/N$-expansion (the genus expansion). The method of orthogonal polynomials and Toda chain.

2. Conformal symmetries: Virasoro conditions and loop equations.

3. Geometry: integrals over moduli spaces, intersection indices, and the Kontsevich matrix model as a tau function of the Korteveg–de Vries hierarchy [7].

4. Generalized Kontsevich model: tau functions of the Kadomtsev–Petviashvili hierarchy and scaling limits.

5. One-matrix model as a Generalized Kontsevich model. The Kontsevich–Penner matrix model.

6. Two-matrix models and higher hierarchies (Toda lattices).

7. Matrix integrals in the limit as $N \to \infty$: the free energy as a tau functions of the Whitham–Krichever hierarchy. Seiberg–Witten equations and associativity equations.

8. Asymptotic expansion in $1/N$: formulating the topological recursion [4].

9. Applications of the topological recursion to various problems of algebraic geometry. [6]

10. Exact relations between different models: cohomological field theories and Givental decomposition [5].

11. Gaussian means, the Kontsevich–Penner matrix model, and discretizations of moduli spaces [2].

12. Enumeration problems in algebraic geometry: Grothendieck dessins d'enfant, Hurwitz numbers and matrix models [3].

The course is supposed to be self-explanatory. By the end of the 4th quarter, lecture notes will be available. No comprehensive textbook exists as yet; for more extensive studies I can recommend review and original papers below and the manuscript of the textbook in preparation (L.Chekhov and A.Mironov) [1] in .pdf, on demand. On June 1-5 there will be a student-oriented workshop in the Lorentz centre, Leiden, the Netherlands, on Quantum Cohomologies and Topological Recursion; the present PhD course can be considered a good preparatory training for this workshop, which will be hopefully attended by a group of QGM students.

References

[1] L. Chekhov and A. Mironov, *Matrix Models: Integrability and Geometry*, manuscript in .pdf, appr. 370 pp.

[2] J.E. Andersen, L.O. Chekhov, P. Norbury, and R.C. Penner, *Models of discretized moduli spaces, cohomological field theories, and Gaussian means*, to appear.

[3] J. Ambjørn and L.O. Chekhov, *The Matrix model for hypergeometric Hurwitz numbers*, arXiv:1409.3553.

[4] L. Chekhov, B. Eynard, and N. Orantin, Free energy topological expansion for the 2-matrix model, *JHEP* **12**(2006)053; hep-th/0603003.

[5] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, arXiv:1110.2949, 38pp.

[6] Eynard, Bertrand and Orantin, Nicolas *Invariants of algebraic curves and topological expansion.* Communications in Number Theory and Physics **1** (2007), 347–452.

[7] M. L. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.*, **147** (1992) 1–23.