Superconformal simple type and Witten's conjecture on the relation between Donaldson and Seiberg-Witten invariants

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Outline

Introduction

Main results

Main results I

For a closed four-manifold X we will use the characteristic numbers,

$$
\begin{aligned}
c_1^2(X) &:= 2e(X) + 3\sigma(X), \\
\chi_h(X) &:= (e(X) + \sigma(X))/4, \\
c(X) &:= \chi_h(X) - c_1^2(X).
\end{aligned} \tag{1}
$$

where $e(X)$ and $\sigma(X)$ are the Euler characteristic and signature of X.

We call a four-manifold *standard* if it is closed, connected, oriented, and smooth with $b^+(X) \geq 3$ and odd and $b^1(X) = 0$.

Main results II

For a standard four-manifold, the Seiberg-Witten invariants define a function,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

on the set of spin^c structures on X.

The *Seiberg-Witten basic classes*, $B(X)$, are the image under $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ of the support of SW_X.

The manifold X has *Seiberg-Witten simple type* if $K^2 = c_1^2(X)$ for all $K \in B(X)$.

Further definitions of and notations for the Donaldson and Seiberg-Witten invariants appear later in this presentation and in [8, 6].

Main results III

Conjecture 1.1 (Witten's conjecture)

Let X be a standard four-manifold. If X has Seiberg-Witten simple type, then X has Kronheimer-Mrowka simple type, the Seiberg-Witten and Kronheimer-Mrowka basic classes coincide, and for any $w \in H^2(X;\mathbb{Z})$ the Donaldson invariants satisfy

$$(2) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}.$$

Main results IV

As defined by Mariño, Moore, and Peradze, [26, 25], the manifold X has *superconformal simple type* if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X;\mathbb{Z})$ characteristic,

$$SW_X^{w,i}(h) = 0 \quad \text{for } i \leq c(X) - 4, \tag{3}$$

and all $h \in H_2(X;\mathbb{R})$, where

$$SW_X^{w,i}(h) := \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) \langle c_1(\mathfrak{s}), h\rangle^i.$$

From [8], we have the

Main results V

Theorem 1.2 (Superconformal simple type $\implies$ Witten's Conjecture holds for all standard four-manifolds)

[8, Main Theorem] Assume that Conjecture 6.7.1 in [5] holds. If a standard four-manifold has superconformal simple type, then it satisfies Witten's Conjecture 1.1.

On the other hand, from [6], we have the

Main results VI

Theorem 1.3 (All standard four-manifolds have superconformal simple type)

[6, Main Theorem] Assume that Conjecture 6.7.1 in [5] holds. If X is a standard four-manifold of Seiberg-Witten simple type, then X has superconformal simple type.

Combining Theorems 1.2 and 1.3 yields the following

Main results VII

Corollary 1.4 (Witten's Conjecture holds for all standard four-manifolds)

Assume that Conjecture 6.7.1 in [5] holds. If X is a standard four-manifold of Seiberg-Witten simple type then X satisfies Witten's Conjecture 1.1.

History of the conjectures

History of the conjectures I

When defining the Seiberg-Witten invariants in [36], Witten also gave a quantum field theory argument yielding the relation in Conjecture 1.1.

Soon after, Pidstrigach and Tyurin [32] proposed the moduli space of SO(3) monopoles as a means to give a mathematically rigourous proof of this conjecture.

In [5], we used the moduli space of SO(3) monopoles to prove — modulo the assumption of certain properties of local SO(3) monopole gluing maps (see [5, Conjecture 6.7.1] and [9, Remark 3.3]) — the SO(3) *monopole cobordism formula* (Theorem 2.6).

The SO(3) monopole cobordism formula gives a relation between the Donaldson and Seiberg-Witten invariants similar to Witten's

History of the conjectures II

Conjecture 1.1, but contains a number of undetermined universal coefficients.

In [11, 12] we computed some of these coefficients directly while in [9] we computed more by comparison with known examples.

Although our previous computations showed that Theorem 2.6 implied Conjecture 1.1 for a wide range of standard four-manifolds, they did not suffice for all.

In our new article [8], we build on our previous methods in [9] to show that the coefficients not determined in [9, Proposition 4.8] are polynomials in one of the parameters on which they depend.

By combining this polynomial dependence with the vanishing condition in the definition of superconformal simple type (3), we

History of the conjectures III

can show that the sum over the terms in the cobordism formula containing these unknown coefficients vanishes.

Hence, the coefficients computed in our previous article [9, Proposition 4.8] suffice to fully determine the Donaldson invariant in terms of Seiberg-Witten invariants and prove Conjecture 1.1.

Proofs of Conjecture 1.1 for restricted classes of standard four-manifolds have appeared elsewhere.

For example, [15], Fintushel and Stern proved Conjecture 1.1 for elliptic surfaces and their blow-ups and rational blow-downs.

Kronheimer and Mrowka in [22, Corollary 7] proved that the cobordism formula in Theorem 2.6 implied Conjecture 1.1 for standard four-manifolds with a tight surface with positive

History of the conjectures IV

self-intersection, a sphere with self-intersection (-1), and Euler number and signature equal to that of a smooth hypersurface in $\mathbb{CP}^3$ of even degree at least six.

In our previous article [9], we generalized the result of Kronheimer-Mrowka to standard four-manifolds of Seiberg-Witten simple type satisfying $c(X) \leq 3$ or which are *abundant* in the sense that $B(X)^\perp \subset H^2(X;\mathbb{Z})$, the orthogonal complement of the basic classes with respect to the intersection form, contained a hyperbolic summand.

We note that by [10, Section A.2], all simply-connected, closed, complex surfaces with $b^+ \geq 3$ are abundant.

History of the conjectures V

The proof in [9] that the SO(3) monopole cobordism formula implies Witten's conjecture used the result of [3] that abundant four-manifolds have superconformal simple type.

In this article, we prove that Theorem 2.6 implies Conjecture 1.1 directly from the superconformal simple type condition.

The examples of non-abundant four-manifolds given in [3] (following [17], one takes log transforms on tori in three disjoint nuclei of a K3 surface) show that there are non-abundant four-manifolds which still satisfy the superconformal simple type condition.

Hence, the results obtained here are stronger than those in [3].

History of the conjectures VI

In [25, 26], Mariño, Moore, and Peradze originally defined the concept of superconformal simple type in the context of supersymmetric quantum field theory and, within that framework, showed that a four-manifold satisfying the superconformal simple type condition obeys the vanishing condition (3).

They conjectured (see [26, Conjecture 7.8.1]) that all standard four-manifolds of Seiberg-Witten simple type obey (3).

Not only do all known examples of standard four-manifolds satisfy (3) (see [26, Section 7]) but the condition is preserved under the standard surgery operations (blow-up, torus sum, and rational blow-down) used to construct new examples.

History of the conjectures VII

Using (3) as a definition of superconformal simple type, they rigorously derived a lower bound on the number of basic classes for manifolds of superconformal simple type (see [26, Theorem 8.1.1]) in terms of topological invariants of the manifold.

Hence, the condition of superconformal simple type is not only of interest to physicists but has important mathematical implications as evidenced by [26, Theorem 8.1.1] and Theorem 1.2.

Finally, we note that the results of [6] use a variant of the SO(3)-monopole cobordism formula to prove that if X is a standard four-manifold of Seiberg-Witten simple type, then X has superconformal simple type.

History of the conjectures VIII

Combining this result with Theorem 1.2 gives Corollary 1.4 which completes this part of the SO(3)-monopole program.

Preliminaries

Seiberg-Witten invariants

Seiberg-Witten invariants I

Detailed expositions of the theory of Seiberg-Witten invariants, introduced by Witten in [36], are provided in [23, 28, 31].

These invariants define an integer-valued map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

on the set of spin^c structures on X.

A spin^c structure, $\mathfrak{s} = (W^\pm, \rho_W)$ on X, consists of a pair of complex rank-two bundles $W^\pm \to X$ and a Clifford multiplication map $\rho : T^*X \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$.

If $\mathfrak{s} \in \mathrm{Spin}^c(X)$, then $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ is characteristic.

Seiberg-Witten invariants II

One calls $c_1(\mathfrak{s})$ a *Seiberg-Witten basic class* if $SW_X(\mathfrak{s}) \neq 0$. Define

(4)
$$B(X) = \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}.$$

If $H^2(X;\mathbb{Z})$ has 2-torsion, then $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ is not injective.

Because we will work with functions involving real homology and cohomology, we define

(5)
$$SW'_X : H^2(X;\mathbb{Z}) \ni K \mapsto \sum_{\mathfrak{s} \in c_1^{-1}(K)} SW_X(\mathfrak{s}) \in \mathbb{Z}.$$

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Seiberg-Witten invariants III

With the preceding definition, Witten's Formula (2) is equivalent to

$$(6)\quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2+K\cdot w)} SW_X'(K) e^{\langle K,h\rangle}.$$

A four-manifold, X, has *Seiberg-Witten simple type* if $SW_X(\mathfrak{s}) \neq 0$ implies that $c_1^2(\mathfrak{s}) = c_1^2(X)$.

As discussed in [28, Section 6.8], there is an involution on $\mathrm{Spin}^c(X)$, denoted by $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$ and defined essentially by taking the complex conjugate vector bundles, and having the property that $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$.

Seiberg-Witten invariants IV

By [28, Corollary 6.8.4], one has $SW_X(\bar{\mathfrak{s}}) = (-1)^{\chi_h(X)} SW_X(\mathfrak{s})$ and so $B(X)$ is closed under the action of $\{\pm 1\}$ on $H^2(X;\mathbb{Z})$.

Versions of the following result have appeared in [14], [16, Theorem 14.1.1], and [31, Theorem 4.6.7].

Seiberg-Witten invariants V

Theorem 2.1 (Blow-up formula for Seiberg-Witten invariants)

[16, Theorem 14.1.1] Let X be a standard four-manifold and let $\widetilde{X} = X\#\overline{\mathbb{CP}}^2$ be its blow-up. Then $\widetilde{X}$ has Seiberg-Witten simple type if and only if that is true for X. If X has Seiberg-Witten simple type, then

$$B(\widetilde{X}) = \{K \pm e^* : K \in B(X)\}, \tag{7}$$

where $e^* \in H^2(\widetilde{X};\mathbb{Z})$ is the Poincaré dual of the exceptional curve, and if $K \in B(X)$, then

$$SW'_{\widetilde{X}}(K \pm e^*) = SW'_X(K).$$

Donaldson invariants

Donaldson invariants I

In [21, Section 2], Kronheimer and Mrowka defined the Donaldson series which encodes the Donaldson invariants developed in [1].

For $w \in H^2(X;\mathbb{Z})$, the *Donaldson invariant* is a linear function,

$$D_X^w : \mathbb{A}(X) \to \mathbb{R},$$

where $\mathbb{A}(X) = \mathrm{Sym}(H_{\mathrm{even}}(X;\mathbb{R}))$, the symmetric algebra.

For $h \in H_2(X;\mathbb{R})$ and a generator $x \in H_0(X;\mathbb{Z})$, we define $D_X^w(h^{\delta-2m}x^m) = 0$ unless

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4. \tag{8}$$

Donaldson invariants II

A four-manifold has *Kronheimer-Mrowka simple type* if for all $w \in H^2(X;\mathbb{Z})$ and all $z \in \mathbb{A}(X)$ one has

$$D_X^w(x^2 z) = 4D_X^w(z). \tag{9}$$

This equality implies that the Donaldson invariants are determined by the *Donaldson series*, the formal power series

$$\mathbf{D}_X^w(h) = D_X^w((1+\tfrac{1}{2}x)e^h), \quad h \in H_2(X;\mathbb{R}). \tag{10}$$

The following result allows us to use a convenient choice of w:

Donaldson invariants III

Proposition 2.2

[9, Proposition 2.5] Let X be a standard four-manifold of Seiberg-Witten simple type. If Witten's Conjecture 1.1 holds for one $w \in H^2(X;\mathbb{Z})$, then it holds for all $w \in H^2(X;\mathbb{Z})$.

The result below allows us to replace a manifold by its blow-up without loss of generality.

Theorem 2.3

[15, Theorem 8.9] Let X be a standard four-manifold. Then Witten's Conjecture 1.1 holds for X if and only if it holds for the blow-up, $\widetilde{X}$.

Witten's conjecture

Witten's conjecture I

It will be more convenient to have Witten's Conjecture 1.1 expressed at the level of the polynomial invariants rather than the power series they form.

Let $B'(X)$ be a fundamental domain for the action of $\{\pm 1\}$ on $B(X)$.

Witten's conjecture II

Lemma 2.4

[9, Lemma 4.2] Let X be a standard four-manifold. Then X satisfies equation (2) and has Kronheimer-Mrowka simple type if and only if the Donaldson invariants of X satisfy $D_X^w(h^{\delta-2m}x^m) = 0$ for $\delta \not\equiv -w^2 - 3\chi_h \pmod 4$ and for $\delta \equiv -w^2 - 3\chi_h \pmod 4$ satisfy

$$(11) \quad D_X^w(h^{\delta-2m}x^m) = \sum_{\substack{i+2k\\=\delta-2m}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)}\nu(K) \times \frac{SW_X'(K)(\delta-2m)!}{2^{k+c(X)-3-m}k!i!}\langle K,h\rangle^i Q_X(h)^k,$$

Witten's conjecture III

Lemma 2.4

where

(12) $$\varepsilon(w, K) := \frac{1}{2}(w^2 + w \cdot K),$$

and

(13) $$\nu(K) = \begin{cases} \frac{1}{2} & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases}$$

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The superconformal simple type property

The superconformal simple type property I

A standard four-manifold X has *superconformal simple type* if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X;\mathbb{Z})$ characteristic and all $h \in H_2(X;\mathbb{R})$,

$$SW_X^{w,i}(h) = 0 \quad \text{for } i \leq c(X) - 4, \tag{14}$$

where

$$SW_X^{w,i}(h) := \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW_X'(K) \langle K, h \rangle^i.$$

Observe that we have rewritten (3) as a sum over $B(X)$ using the expression (5). We further note that the property (14) is invariant under blow-up.

The superconformal simple type property II

Lemma 2.5

[26, Theorem 7.3.1], [6, Lemma 6.1] A standard manifold, X, has superconformal simple type if and only if its blow-up, $\widetilde{X}$, has superconformal simple type.

SO(3) monopoles and Witten's conjecture

SO(3) monopoles and Witten's conjecture I

The SO(3)-monopole cobordism formula given below provides an expression for the Donaldson invariant in terms of the Seiberg-Witten invariants.

SO(3) monopoles and Witten's conjecture II

Theorem 2.6 (SO(3)-monopole cobordism formula)

[5, Main Theorem] Let X be a standard four-manifold of Seiberg-Witten simple type. Assume that [5, Conjecture 6.7.1] holds. Assume further that $w, \Lambda \in H^2(X;\mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfy

$$w - \Lambda \equiv w_2(X) \pmod 2, \tag{15a}$$

$$I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X) > \delta, \tag{15b}$$

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4, \tag{15c}$$

$$\delta - 2m \geq 0. \tag{15d}$$

Then, for any $h \in H_2(X;\mathbb{R})$ and positive generator $x \in H_0(X;\mathbb{Z})$, we have

SO(3) monopoles and Witten's conjecture III

Theorem 2.6 (SO(3)-monopole cobordism formula)

(16)

$$D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-\Lambda)\cdot K)} SW_X'(K) \times f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h),$$

where the map,

$$f_{\delta,m}(h) : \mathbb{Z} \times \mathbb{Z} \times H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{R}[h],$$

takes values in the ring of polynomials in the variable h with

SO(3) monopoles and Witten's conjecture IV

Theorem 2.6 (SO(3)-monopole cobordism formula)

real coefficients, is universal (independent of X) and is given by

$$(17)\quad f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h) := \sum_{\substack{i+j+2k\\=\delta-2m}} a_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m) \times \langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k.$$

SO(3) monopoles and Witten's conjecture V

Theorem 2.6 (SO(3)-monopole cobordism formula)

For each triple, $i, j, k \in \mathbb{N}$, the coefficients,

$$a_{i,j,k} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \to \mathbb{R},$$

are universal (independent of X) real analytic functions of the variables $\chi_h(X)$, $c_1^2(X)$, $c_1(\mathfrak{s}) \cdot \Lambda$, Λ^2, and m.

Remark 2.7

Theorem 2.6 depends on the expected analytical result [5, Conjecture 6.7.1], on which work is in progress. The analytical issues are summarized in [9, Remark 3.3].

SO(3) monopoles and Witten's conjecture VI

The SO(3)-*monopole equations* take the form,

$$(18)\qquad \begin{aligned} F_A^+ - \rho^{-1}(\Phi \otimes \Phi^*)_{00} &= 0, \\ D_A\Phi &= 0, \end{aligned}$$

where A is a unitary connection on a Hermitian, rank-two vector bundle, E, and $\Phi \in \Omega^0(X; W^+ \otimes E)$.

Theorem 2.6 is proved with the aid of the moduli space, $\mathscr{M}_{\mathfrak{t}}$, of solutions to (18) moduli gauge-equivalence, where $\mathfrak{t} = (\rho, W^{\pm}, E)$.

The space, $\mathscr{M}_{\mathfrak{t}}$, contains the moduli space of *anti-self-dual connections*, M_κ^w, and moduli spaces, $M_{\mathfrak{s}}$, of Seiberg-Witten monopoles where the connections, A, on E becomes reducible with respect to different splittings, $E = L_1 \oplus L_2$.

SO(3) monopoles and Witten's conjecture VII

The space, $\mathscr{M}_{\mathfrak{t}}$, admits an Uhlenbeck compactification, $\bar{\mathscr{M}}_{\mathfrak{t}}$, and the lower strate of $\bar{\mathscr{M}}_{\mathfrak{t}}$ contain additional Seiberg-Witten moduli spaces.

The cobordism formula (16) is proved by pairing a cup product of suitable cohomology classes on $\bar{\mathscr{M}}_{\mathfrak{t}}$ with the link of M_{κ}^{w}, giving rise to multiples of the Donaldson invariant, and links of the Seiberg-Witten moduli spaces, $M_{\mathfrak{s}}$.

Key ideas in proof that superconformal simple type $\implies$ Witten's Conjecture

Key ideas in proof that SCST $\implies$ WC I

The proof of Theorem 1.2 relies on a combination of the ingredients mentioned below.

Blow-up formula for Donaldson invariants

Let $\widetilde{X} \to X$ be the blow-up of X at one point, let $e \in H_2(\widetilde{X};\mathbb{Z})$ be the fundamental class of the exceptional curve, and let $e^* \in H^2(\widetilde{X};\mathbb{Z})$ be the Poincaré dual of e.

Using the direct sum decomposition of the homology and cohomology of $\widetilde{X}$, we can consider both the homology and cohomology of X as subspaces of those of $\widetilde{X}$.

Denote $\tilde{w} := w + e^*$. The blow-up formula [19, 24] gives

$$D_X^w(h^{\delta-2m}x^m) = D_{\widetilde{X}}^{\tilde{w}}(h^{\delta-2m}ex^m). \tag{19}$$

Key ideas in proof that SCST $\Longrightarrow$ WC II

Blow-up formula for Seiberg-Witten invariants

By Theorem 2.1,

$$B'(\widetilde{X}) = \{K_\varphi = K + (-1)^\varphi e^* : K \in B'(X),\ \varphi \in \mathbb{Z}/2\mathbb{Z}\}. \tag{20}$$

Difference equation for $b_{i,j,k}$

As a consequence of the superconformal type property of X, we show that the coefficients $a_{i,j,k}$ which are not determined by [9, Proposition 4.8], satisfy a *difference equation* in the parameter $K \cdot \Lambda$ and thus can be written as a polynomial in this parameter.

The Fintushel-Park-Stern family of example manifolds

Key ideas in proof that SCST $\implies$ WC III

In [9, Section 4.2], we used the manifolds constructed by Fintushel, Park and Stern in [13] to give a family of standard four-manifolds, X_q, for $q = 2, 3, \dots,$ obeying the following conditions:

1. X_q satisfies Witten's Conjecture 1.1;
2. For $q = 2, 3, \dots,$ one has $\chi_h(X_q) = q$ and $c(X_q) = 3$;
3. $B'(X_q) = \{K\}$ with $K \neq 0$;
4. For each q, there are classes $f_1, f_2 \in H^2(X_q; \mathbb{Z})$ satisfying

$$f_1 \cdot f_2 = 1, \quad f_i^2 = 0, \quad f_i \cdot K = 0 \quad \text{for} \quad i = 1, 2, \tag{21a}$$

$$\{f_1, f_2, K\} \text{ is linearly independent in } H^2(X_q; \mathbb{R}), \tag{21b}$$

$$Q_{X_q} \restriction \operatorname{Ker} f_1 \cap \operatorname{Ker} f_2 \cap \operatorname{Ker} K \text{ is non-zero.} \tag{21c}$$

Key ideas in proof that SCST $\implies$ WC IV

Let $X_q(n)$ be the blow-up of X_q at n points,

$$X_q(n) := X_q \underbrace{\#\overline{\mathbb{CP}}^2 \cdots \#\overline{\mathbb{CP}}^2}_{n \text{ copies}}. \tag{22}$$

Then $X_q(n)$ is a standard four-manifold of Seiberg-Witten simple type and satisfies Witten's Conjecture 1.1 by Theorem 2.3, with

$$\chi_h(X_q(n)) = q, \quad c_1^2(X_q(n)) = q-n-3, \quad c(X_q(n)) = n+3. \tag{23}$$

Key ideas in proof that all 4-manifolds are SCST

Key ideas in proof that all 4-manifolds are SCST I

We exploit the fact that the SO(3)-monopole cobordism provides additional relations among the Seiberg-Witten invariants than just the relation (16) in Theorem 2.6 which computes values of Donaldson invariants in terms of Seiberg-Witten invariants.

There is a variant of the SO(3)-monopole cobordism formula (16) where the pairing with the link of the moduli space of anti-self-dual connections vanishes by a dimension-counting argument.

This variant of the SO(3)-monopole cobordism formula states that a sum over $K \in B(X)$ of pairings with links of the Seiberg-Witten

Key ideas in proof that all 4-manifolds are SCST II

moduli space corresponding to K vanishes, giving an equality of the form

$$0 = \sum_{k=0}^{\ell} a_{c-2v+2k,0,\ell-k} SW_X^{w,c-2v+2k} Q_X^{\ell-k}, \tag{24}$$

where we abbreviate $c = c(X)$.

We then show that the coefficient $a_{c-2v,0,\ell}$ appearing in (24) is non-zero by applying the methods used by Kotschick and Morgan [20] to the topological description of the link of the Seiberg-Witten moduli space given in our article [5].

Next, we show that the coefficients $a_{c-2v+2k,0,\ell-k}$ in (24) vanish if $c - 2v + 2k \geq c - 3$.

Key ideas in proof that all 4-manifolds are SCST III

Finally, we combine this information on the coefficients and give an inductive argument proving Theorem 1.3.

And Conjecture 6.7.1?

And Conjecture 6.17? I

We need to explain a little more about Conjecture 6.7.1, from our book [5], where Theorem 2.6 is proved.

The proof of Theorem 2.6 (the SO(3)-monopole cobordism formula) assumes the hypothesis [5, Conjecture 6.7.1] that the *local gluing map* for a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ gives a continuous parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$, for each smooth stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$.

These local gluing maps are the analogues for SO(3) monopoles of the local gluing maps for anti-self-dual SO(3) connections constructed by Taubes in [33, 34, 35], Donaldson and Kronheimer in [2, §7.2], and Morgan and Mrowka [29, 30].

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And Conjecture 6.17? II

We have established the *existence* of local gluing maps in our article [7].

We expect that a proof of the *continuity* for the local gluing maps with respect to Uhlenbeck limits should be similar to our proof in [4] of this property for the local gluing maps for anti-self-dual SO(3) connections.

The remaining properties of local gluing maps assumed in [5] are that they are *injective*, and also *surjective* in the sense that elements of $\bar{\mathscr{M}}_{\mathfrak{t}}$ sufficiently close (in the Uhlenbeck topology) to $M_{\mathfrak{s}} \times \Sigma$ are in the image of at least one of the local gluing maps.

In special cases, proofs of these properties for the local gluing maps for anti-self-dual SO(3) connections (namely, continuity with

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And Conjecture 6.17? III

respect to Uhlenbeck limits, injectivity, and surjectivity) have been given in [2, §7.2.5, 7.2.6], [33, 34, 35].

The authors are currently developing a proof of the required properties for the local gluing maps for SO(3) monopoles in a book in progress.

Our proof will also yield the analogous properties for the local gluing maps for anti-self-dual SO(3) connections.

Thank you for your attention!

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