The Calabi-Yau Landscape

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幾何、 量子拓撲 與 漸進分析； 2018六月

A Classic Problem in Mathematics

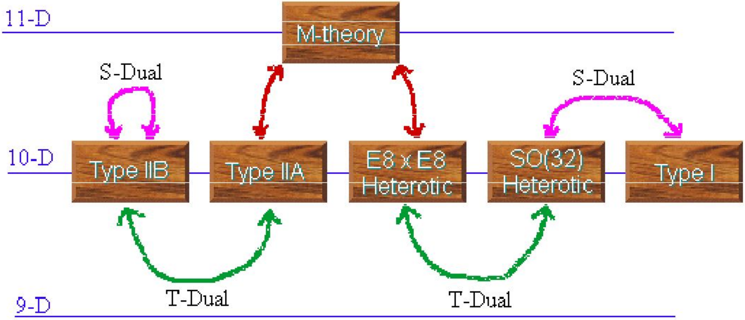
- Relation between curvature (differential geometry) and characteristic classes (algebraic geometry)?
- CONJECTURE [E. Calabi, 1954, 1957]: M compact Kähler manifold (g, ω) and $([R] = [c_1(M)])_{H^{1,1}(M)}$. Geometric Nomenclature
  Then $\exists!(\tilde{g}, \tilde{\omega})$ such that $([\omega] = [\tilde{\omega}])_{H^2(M;\mathbb{R})}$ and $Ricci(\tilde{\omega}) = R$.

Rmk: $c_1(M) = 0 \Leftrightarrow$ Ricci-flat

- THEOREM [S-T Yau, 1977-8; Fields 1982] Calabi-Yau: Kähler and Ricci-flat
- Important example: $\dim_{\mathbb{C}} = 1$, T^2 (elliptic curve)

A Opportune Development in Physics

String Theory:



The most important equation: $\boxed{10 = 4 + 6}$

String Phenomenology

- Superstring: unifies QM + GR in 10 dimensions: X^{10}
- We live in some M^4 (assume maximally symmetric)

$$R_{\mu\nu\rho\lambda} = \frac{R}{12}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}), \qquad R \begin{cases} = 0 & \text{Minkowski} \\ > 0 & \text{de Sitter (dS)} \\ < 0 & \text{anti-de Sitter (AdS)} \end{cases}$$

- $10 = 4 + 6$: **two scenarios**
  1. SMALL: compactification $X^{10} \simeq M^4 \times X^6$
  2. LARGE: brane-world trapped on a 3-brane in 10-D
- want: supersymmetry at intermediate scale (between string and EW)
- want: classical vacuum of string theory on X^{10} preserves $\mathcal{N} = 1$ SUSY in M^4

Heterotic Compactification

[Candelas-Horowitz-Strominger-Witten] (1986): $\delta_{SUSY} S_{Het} = 0$

- $S \sim \int d^{10}x\sqrt{g}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3'|^2) - \frac{1}{g_s^2}\,\mathrm{Tr}\,|F_2|^2\right]$+SUSY)

gravitino	$\delta_\epsilon\Psi_{M=1,\ldots,10} = \nabla_M\epsilon - \frac{1}{4}H_M^{(3)}\epsilon$
dilatino	$\delta_\epsilon\lambda = -\frac{1}{2}\Gamma^M\partial_M\Phi\;\epsilon + \frac{1}{4}H_M^{(3)}\epsilon$
adjoint YM	$\delta_\epsilon\chi = -\frac{1}{2}F^{(2)}\epsilon$
Bianchi	$dH^{(3)} = \frac{\alpha'}{4}[\mathrm{Tr}(R\wedge R) - \mathrm{Tr}(F\wedge F)]$

- Assume $H^{(3)} = 0$ (can generalise) $\rightsquigarrow$ Killing spinor equation:

  $\delta_\epsilon\Psi_{M=1,\ldots,10} = \nabla_M\epsilon = 0 = \nabla_M\xi(x^{\mu=1,\ldots,4})\eta(y^{m=1,\ldots,6})$

- External 4D Space: $[\nabla_\mu,\nabla_\nu]\xi(x) = \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\xi(x) = 0 \rightsquigarrow R = 0 \Rightarrow M$ is Minkowski (actually the universe is now believed to be dS)

- Internal 6D Space: $R_{mn} = 0$ (but not necessarily max symmetric)

Mille Viæ ducunt homines Romam ...

- X^6 as a spin 6-manifold: holonomy group is $SO(6) \simeq SU(4)$
  - want covariant constant spinor: largest possible is $SU(4) \to SU(3)$ with $4 \to 3 \oplus 1 \Rightarrow X^6$ has $SU(3)$ holonomy
  - Thus $\epsilon(x^{1,\ldots,4}, y^{1,\ldots,6}) = \xi_+ \otimes \eta_+(y) + \xi_- \otimes \eta_-(y)$ with $\eta_+^* = \eta_-$ and ξ constant
- Define $J_m^n = i\eta_+^\dagger \gamma_m^n \eta_+ = -i\eta_-^\dagger \gamma_m^n \eta_-$, check: $J_m^n J_n^p = -\delta_m^n$
- Can show X^6 is a Kähler manifold of $\dim_{\mathbb{C}} = 3$, with $SU(3)$ holonomy
- Three other SUSY variation equations (recall $H^{(3)} = 0$ by choice)
  - choose constant dilation $\Phi \rightsquigarrow \delta_\epsilon = 0$
  - choose $R = F$ (spin connection for gauge field): Bianchi satisfied
  - Also $R = 0$ so $\delta_\epsilon \chi = 0$

Special Holonomy

- For a Riemannian, spin manifold M of real dimension d, holonomy is $Spin(d)$ as double cover of $SO(d)$ *generically*, but could have *special holonomy*

Holonomy $\mathcal{H} \subset$	Manifold Type (IFF)
$U(d/2)$	Kähler
$SU(d/2)$	Calabi-Yau
$Sp(d/4)$	Hyper-Kähler
$Sp(d/4) \times Sp(1)$	Quaternionic-Kähler

- X^6 **is Calabi-Yau**
- no-where vanishing holomorphic 3-form: $\Omega^{(3,0)} = \frac{1}{3!}\Omega_{mnp} dz^m \wedge dz^n \wedge dz^p$

  with $\Omega_{mnp} := \eta_-^T \ \gamma^{[m}\gamma^n\gamma^{p]} \ \eta_-$

  check: $d\Omega = 0$ but not exact; $\Omega \wedge \bar{\Omega} \sim$ Volume form

Summary

Some equivalent Definitions for X^6 Calabi-Yau Threefold

- Kähler, $c_1(TX) = 0$
- Kähler, vanishing Ricci curvature
- Kähler, holonomy $\subset SU(n)$
- Kähler, nowhere vanishing global holomorphic 3-form (volume)
- Covariant constant spinor
- Canonical bundle (sheaf) $K_X := \bigwedge^n T_X^* \simeq \mathcal{O}_X$
- low-energy SUSY in 4D from string compactification

Some Topological Properties I

- **Hodge Numbers** $h^{p,q}(X) = \dim H^{p,q}_{\bar{\partial}}(X)$
  - Hodge decomposition and Betti Numbers: $b_k = \sum\limits_{p+q=k} h^{p,q}(X)$
  - Complex conjugation $\rightsquigarrow h^{p,q} = h^{q,p}$
  - Hodge star (Poincaré) $\rightsquigarrow h^{p,q} = h^{n-p,n-q}$
  - Hodge Diamond:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{0,1} & & h^{0,1} & & \\ & h^{0,2} & & h^{1,1} & & h^{0,2} & \\ h^{0,3} & & h^{2,1} & & h^{2,1} & & h^{0,3} \\ & h^{0,2} & & h^{1,1} & & h^{0,2} & \\ & & h^{0,1} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

  - Compact, connected, Kähler: $h^{0,0} = 1$ (constant functions)
  - If simply-connected:

    $\pi_1(X) = 0 \rightsquigarrow H_1(X) = \pi_1(X)/[\ ,\ ] = 0 \rightsquigarrow h^{1,0} = h^{0,1} = 0$

Some Topological Properties II

- Finally, CY3 has $h^{3,0} = h^{0,3} = 1$ [unique holomorphic 3-form], also $h^{p,0} = h^{3-p,0}$ by contracting $(p,0)$-form with $\bar{\Omega}$ to give $(p,3)$-form, then use Poincaré duality to give $(3-p,0)$-form
- 2-topological numbers for a (connected, simply connected) CY3:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(Kähler, Complex-Structure) : $(h^{1,1}, h^{2,1})$

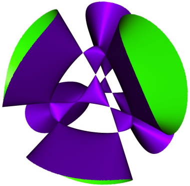
$\chi(X) = 2(h^{1,1} - h^{2,1})$

- Moduli Space of CY3 locally: $\mathcal{M} \simeq \mathcal{M}^{2,1} \times \mathcal{M}^{1,1}$

Explicit Examples of Calabi-Yau Manifolds

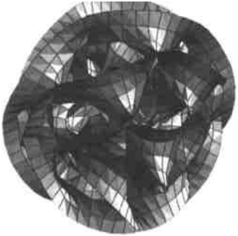
- $d = 1$ Torus $T^2 = S^1 \times S^1$

- $d = 2$ K3



; 4-torus: $T^4 = \left(S^1\right)^4$

- $d = 3$ CY3: Unclassified, billions known



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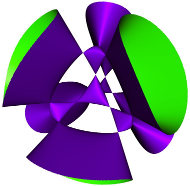
- $d = 2$ K3  ; 4-torus: $T^4 = \left(S^1\right)^4$
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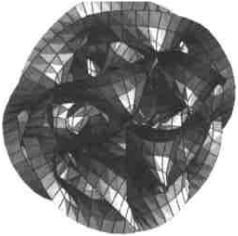

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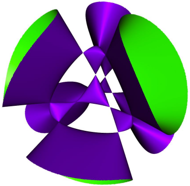
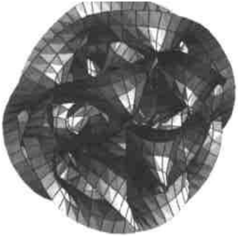
- $d = 2$ K3  ; 4-torus: $T^4 = \left(S^1\right)^4$

- $d = 3$ CY3: Unclassified, billions known 

As Projective Varieties

- Embed X into $\mathbb{P}^n$ as **complete intersection** of K polynomials

$$n = K + 3$$

- Canonical bundle $\mathcal{K}_X \simeq \wedge^{\dim(X)} T_X^*$; algebraic condition for Calabi-Yau: $K_X \simeq \mathcal{O}_X$ (indeed $c_1(TX) = 0$)
- **Adjunction formula** for subvariety $X \subset A$: $\mathcal{K}_X = (K_A \otimes N^*)|_X$
- Recall $K_{A=\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$ and $K_X \simeq \mathcal{O}_X$, thus:

$$\text{degree}(X) = n + 1$$

- Find only 5 solutions. These all have $h^{1,1}(X) = 1$, inherited from the 1 Kähler class of $\mathbb{P}^n$; called cyclic Calabi-Yau threefolds

Cyclic Manifolds

Intersection	$\mathcal{A}$	Configuration	$\chi(X)$	$h^{1,1}(X)$	$h^{2,1}(X)$	$d(X)$	$\tilde{c}_2(TX)$
Quintic	$\mathbb{P}^4$	$[4\|5]$	-200	1	101	5	10
Quadric and quartic	$\mathbb{P}^5$	$[5\|2\ 4]$	-176	1	89	8	7
Two cubics	$\mathbb{P}^5$	$[5\|3\ 3]$	-144	1	73	9	6
Cubic and 2 quadrics	$\mathbb{P}^6$	$[6\|3\ 2\ 2]$	-144	1	73	12	5
Four quadrics	$\mathbb{P}^7$	$[7\|2\ 2\ 2\ 2]$	-128	1	65	16	4

- Euler numbers quite large, $d(X)$ is volume normalisation
- used standard **matrix configuration notation**
- most famous example: Quintic 3-fold $[4|5]$

$$\{\sum_{i=0}^{4} x_i^5 = 0\} \subset \mathbb{P}^4_{[x_0:\ldots x_4]}$$

written as Fermat quintic, also has $h^{2,1}(X) = 101$ deformation parameters

Part I

Strings and the Compact Calabi-Yau Landscape

Triadophilia: A 20-year search

- A 2-decade Problem: [Candelas-Horowitz-Strominger-Witten] (1986)
  - $E_8 \supset SU(3) \times SU(2) \times U(1)$ Natural Gauge Unification
  - Mathematically succinct
  - Witten: "still the best hope for the real world"
- CY3 X, tangent bundle $SU(3) \Rightarrow E_6$ GUT: commutant $E_8 \to SU(3) \times E_6$ (generalize later)
  - Particle Spectrum:

	Generation	$n_{27} = h^1(X, TX) = h^{2,1}_{\bar\partial}(X)$
	Anti-Generation	$n_{\overline{27}} = h^1(X, TX^*) = h^{1,1}_{\bar\partial}(X)$

- Net-generation: $\chi = 2(h^{1,1} - h^{2,1})$
- Question: Are there Calabi-Yau threefolds with Euler character ± 6?

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Complete Intersection Calabi-Yau (CICY) 3-folds

- immediately: Quintic Q in $\mathbb{P}^4$ is CY3, recall: $Q_{\chi}^{h^{1,1},h^{2,1}} = Q_{-200}^{1,101}$ so too may generations (even with quotient $-200 \notin 3\mathbb{Z}$)
- [Candelas-A. He-Hübsch-Lutken-Schimmrigk-Berglund] (1986-1990)
  - dim(Ambient space) - #(defining Eq.) = 3 (complete intersection)

$$M = \left[\begin{array}{c|cccc} n_1 & q_1^1 & q_1^2 & \dots & q_1^K \\ n_2 & q_2^1 & q_2^2 & \dots & q_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_m & q_m^1 & q_m^2 & \dots & q_m^K \end{array}\right]_{m \times K}$$

– K eqns of multi-degree $q_j^i \in \mathbb{Z}_{\geq 0}$ embedded in $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$

– $c_1(X) = 0 \rightsquigarrow \sum\limits_{j=1}^{K} q_r^j = n_r + 1$

– M^T also CICY

  - Famous Examples

The First Data-sets in Mathematical Physics/Geometry I

- Problem: *classify all configuration matrices*; employed the best computers at the time (**CERN supercomputer**)

  q.v. magnetic tape and dot-matrix printout in Philip's office

  - 7890 matrices from 1×1 to max(row) $= 12$, max(col) $= 15$; with $q_j^i \in [0, 5]$
  - 266 distinct Hodge pairs $(h^{1,1}, h^{2,1}) = (1, 65), \ldots, (19, 19)$
  - 70 distinct Euler $\chi \in [-200, 0]$ (all negative)
  - [V. Braun, 1003.3235] : 195 have freely-acting symmetries (quotients), 37 different finite groups (from $\mathbb{Z}_2$ to $\mathbb{Z}_8 \rtimes H_8$)

- Rmk: Integration pulls back to ambient product of projective space A

$$\int_X \cdot = \int_A \mu \wedge \cdot \ , \qquad \mu := \bigwedge_{j=1}^{K} \left(\sum_{r=1}^{m} q_r^j J_r \right) \ .$$

Topological Quantities

- Chern classes of CICY

$$
\begin{aligned}
c_1^r(T_X) &= 0 \\
c_2^{rs}(T_X) &= \frac{1}{2}\left[-\delta^{rs}(n_r+1) + \sum_{j=1}^{K} q_j^r q_j^s\right] \\
c_3^{rst}(T_X) &= \frac{1}{3}\left[\delta^{rst}(n_r+1) - \sum_{j=1}^{K} q_j^r q_j^s q_j^t\right]
\end{aligned}
$$

- Triple intersection numbers: $d_{rst} = \int_X \cdot = \int_A J_r \wedge J_s \wedge J_t$
- Euler number: $\chi(X) = \text{Coefficient}(c_3^{rst} J_r J_s J_t \cdot \mu, \quad \prod_{r=1}^{m} J_r^{n_r})$
- As always, computing individual terms $(h^{1,1}, h^{2,1})$ hard even though $h^{1,1} - h^{2,1} = \frac{1}{2}\chi$ (index theorem)

Computing Hodge Numbers: Sketch

- Recall Hodge decomposition $H^{p,q}(X) \simeq H^q(X, \wedge^p T^\star X) \rightsquigarrow$

$$H^{1,1}(X) = H^1(X, T_X^\star), \qquad H^{2,1}(X) \simeq H^{1,2} = H^2(X, T_X^\star) \simeq H^1(X, T_X)$$

- Euler Sequence for subvariety $X \subset A$ is short exact:

$$0 \to T_X \to T_M|_X \to N_X \to 0$$

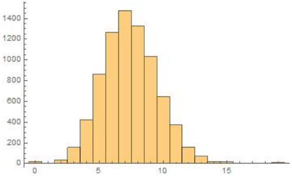
- Induces long exact sequence in cohomology:

$$\begin{array}{ccccccccc}
0 & \to & \cancel{H^0(X, T_X)} \nearrow 0 & \to & H^0(X, T_A|_X) & \to & H^0(X, N_X) & \to \\
 & \to & \boxed{H^1(X, T_X)} & \stackrel{d}{\to} & H^1(X, T_A|_X) & \to & H^1(X, N_X) & \to \\
 & \to & H^2(X, T_X) & \to & \ldots & & &
\end{array}$$

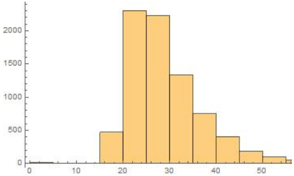
- Need to compute Rk(d), cohomology and $H^i(X, T_A|_X)$ (Cf. Hübsch)

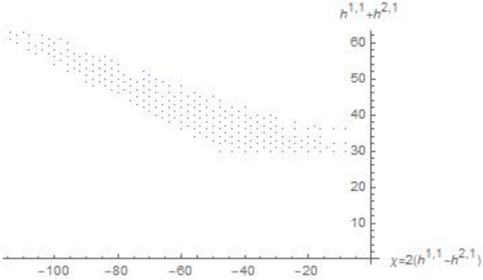
Distribution

$h^{1,1}$



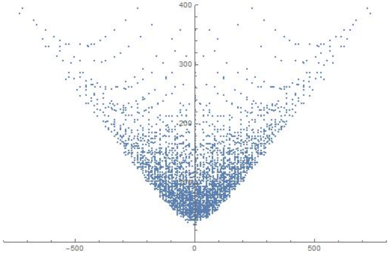
$h^{2,1}$





[Candelas-Lynker-Schimmrigk, 1990] Hypersurfaces in Weighted $\mathbb{P}^4$

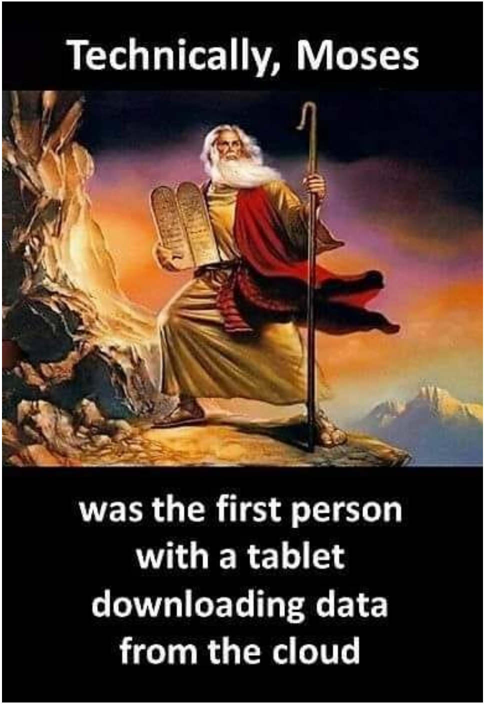
- generic homog deg $= \sum_{i=0}^{4} w_i$ polynomial in $W\mathbb{P}^4_{[w_0:w_1:w_2:w_3:w_4]} \simeq (\mathbb{C}^5 - \{0\})/(x_0, x_1, x_2, x_3, x_4) \sim (\lambda^{w_0} x_0, \lambda^{w_1} x_1, \lambda^{w_2} x_2, \lambda^{w_3} x_3, \lambda^{w_4} x_4)$
- specified by a single integer 5-vector: w_i
- Rmk: ambient WP4 is singular (need to resolve)



7555 inequivalent 5-vectors w_i

2780 Hodge pairs

$\chi \in [-971, 469]$



The age of data science in mathematical physics/string theory not as recent as you might think

Elliptically Fibered CY3: [Gross, Morrison-Vafa, 1994]

- X elliptically fibered over some base B: as Weierstraß model in $\mathbb{P}^2_{[x:y:z]}$-bundle over B (g_2, g_3 complex structure coeff)

$$zy^2 = 4x^3 - g_2 x z^2 - g_3 z^3$$

  x, y, z, g_2, g_3 must be sections of powers of some line bundle $\mathcal{L}$ over B

- Specifically (x, y, z, g_2, g_3) are global sections of $(\mathcal{L}^{\oplus 2}, \mathcal{L}^{\oplus 3}, \mathcal{O}_B, \mathcal{L}^{\oplus 4}, \mathcal{L}^{\oplus 6})$
- $c_1(TX) = 0 \Rightarrow \mathcal{L} \simeq K_B^{-1} \Rightarrow B$ highly constrained :
  1. del Pezzo surface $d\mathbb{P}_{r=1,\ldots,9}$: $\mathbb{P}^2$ blown up at r points
  2. Hirzebruch surface $\mathbb{F}_{r=0,\ldots 12}$: $\mathbb{P}^1$-bundle over $\mathbb{P}^1$
  3. Enriques surface $\mathbb{E}$: involution of K3
  4. Blowups of $\mathbb{F}_r$

Ne Plus Ultra: The Kreuzer-Skarke Dataset

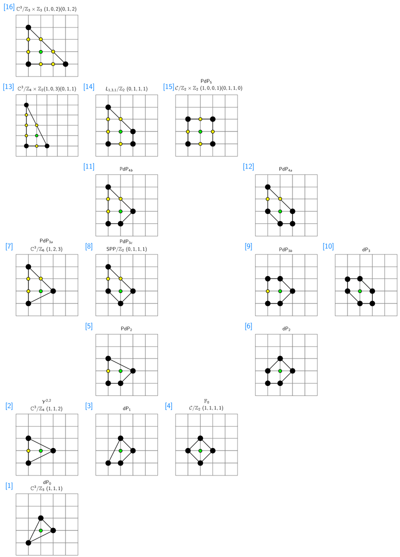
- Generalize WP4, take Toric Variety $A(\Delta_n)$ and consider hypersurface therein
- $A(\Delta_n)$ is special: it is constructed from a reflexive polytope Lattice Polytopes
- THM [Batyrev-Borisov, '90s] anti-canonical divisor in $X(\Delta_n)$ gives a smooth Calabi-Yau $(n-1)$-fold as hypersurface:

$$0 = \sum_{\mathbf{m} \in \Delta} C_{\mathbf{m}} \prod_{\rho=1}^{k} x_\rho^{\langle \mathbf{m}, \mathbf{v}_\rho \rangle + 1} , \qquad \Delta^\circ = \{\mathbf{v} \in \mathbb{R}^4 \mid \langle \mathbf{m}, \mathbf{v} \rangle \geq -1 \ \forall \mathbf{m} \in \Delta\}$$

$\mathbf{v}_\rho$ vertices of Δ.

- Simplest case: $A = \mathbb{P}^4$ and we have quintic $[4|5]$ again.

$$\Delta : \begin{array}{rcl} \mathbf{m}_1 & = & (-1,-1,-1,-1), \\ \mathbf{m}_2 & = & (\ 4,-1,-1,-1), \\ \mathbf{m}_3 & = & (-1,\ 4,-1,-1), \\ \mathbf{m}_4 & = & (-1,-1,\ 4,-1), \\ \mathbf{m}_5 & = & (-1,-1,-1,\ 4)\ , \end{array} \qquad \Delta^\circ : \begin{array}{rcl} \mathbf{v}_1 & = & (1,0,0,0), \\ \mathbf{v}_2 & = & (0,1,0,0), \\ \mathbf{v}_3 & = & (0,0,1,0), \\ \mathbf{v}_4 & = & (0,0,0,1), \\ \mathbf{v}_5 & = & (-1,-1,-1,-1)\ . \end{array}$$



- THM (classical): All Δ_2 are $GL(2;\mathbb{Z})$ equivalent to one of the 16
- $\rightarrow$ #vertices: 3, . . . , 6
- $\uparrow$ #lattice points: 4, . . . , 10
- 4 self-dual
- 5 smooth $X(\Delta_2)$ = toric del Pezzo surfaces: $dP_{0,1,2,3}$, $\mathbb{P}^1 \times \mathbb{P}^1$ (smooth toric Fano surfaces)

Known Classification Results

- $GL(n;\mathbb{Z})$-equivalence classes of reflexive Δ_n finite for each n
- Kreuzer[†]-Skarke (Using PALP) [1990s]: a fascinating sequence

dimension	1	2	3	4	...
# Reflexive Polytopes	1	16	4319	473,800,776	...
# Regular	1	5	18	124	...

- $n \geq 5$ still not classified; generating function also not known
- Smooth ones known for a few more dimensions (Kreuzer-Nill, Øbro, Paffenholz): $\{1,\ 5,\ 18,\ 124,\ 866,\ 7622,\ 72256,\ 749892,\ 8229721\ldots\}$
- $n = 2, 3$ built into SAGE

Tour de Force: Kreuzer-Skarke

- Kreuzer[†]-Skarke 1997-2002: 473,800,776 Δ_4
  - AT LEAST this many CY3 hypersurfaces in $A(\Delta_4)$: CY3 depends on triangulation (resolution) of Δ, but Hodge numbers only depend on Δ_4 (Batyrev-Borisov):

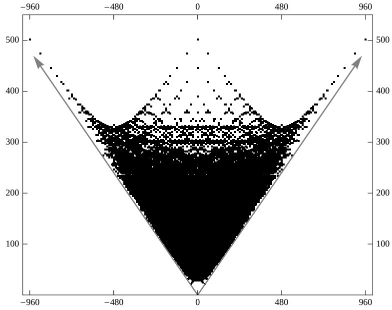
$$\begin{aligned} h^{1,1}(X) &= \ell(\Delta^\circ) - \sum_{\text{codim}\theta^\circ=1} \ell^\circ(\theta^\circ) + \sum_{\text{codim}\theta^\circ=2} \ell^\circ(\theta^\circ)\ell^\circ(\theta) - 5; \\ h^{1,2}(X) &= \ell(\Delta) - \sum_{\text{codim}\theta=1} \ell^\circ(\theta) + \sum_{\text{codim}\theta=2} \ell^\circ(\theta)\ell^\circ(\theta^\circ) - 5 \,. \end{aligned}$$

  - Dual polytope $\Delta \leftrightarrow \Delta^\circ$ = mirror symmetry
- Vienna group (KS, Knapp,...), Oxford group (Candelas, Lukas, YHH, ...), MIT group (Taylor,Johnson, Wang, ...), Northeastern/Wits Collab (Nelson, Jejjala, YHH), Virginia Tech (Anderson, Gray, Lee, ...) Tsinghua/London/Oxford Collab (Yau, Seong, YHH)

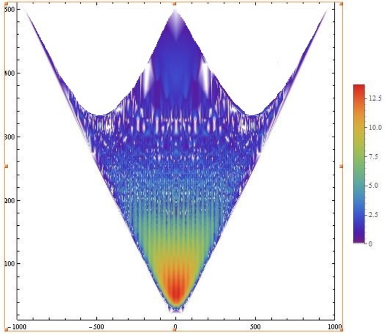
Georgia O'Keefe

30,108 distinct Hodge pairs, $\chi \in [-496, 496]$;

$(h^{1,1}, h^{2,1}) = (27, 27)$ dominates: 910113 instances



In Philip's Office



YHH (1308.0186)

Refined Structure in KS Data

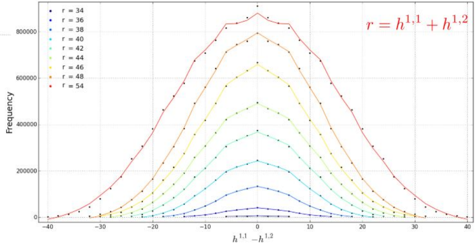
- DATABASES:

  http://hep.itp.tuwien.ac.at/~kreuzer/CY/

  http://www.rossealtman.com/

- Altman-Gray-YHH-Jejjala-Nelson 2014-17 triangulate Δ_4 (orders more than 1/2-billion): up to $h^{1,1} = 7$
- Candelas-Constantin-Davies-Mishra 2011-17 special small Hodge numbers
- Taylor, Johnson, Wang et al. 2012-17 elliptic fibrations
- YHH-Jejjala-Pontiggia 2016 distribution of Hodge, χ, Pseudo-Voigt

KS stats



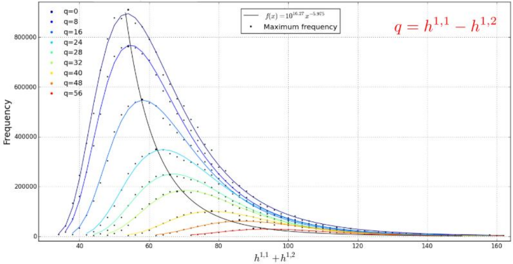
Pseudo-Voigt distribution

sum of Gaussian and Cauchy

$$(1-\alpha)\frac{A}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \alpha\frac{A}{\pi}\left[\frac{\sigma^2}{(x-\mu)^2+\sigma^2}\right]$$

Planck distribution

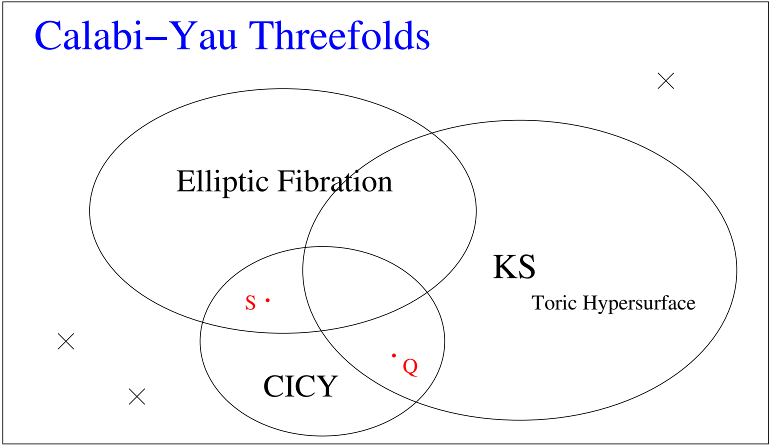
$$\frac{A}{x^n}\frac{1}{e^{b/(x-c)}-1}$$



He, VJ, Pontiggia (2015)

The Compact CY3 Landscape

- 20 years of research by mathematicians and physicists
- 10^{10} million data-points (and growing)



CY3 Compactification: Recent Development

- E_6 GUTs less favourable, $SU(5)$ and $SO(10)$ GUTs: general embedding
  - Instead of TX, use (poly-)stable holomorphic vector bundle V
  - LE particles $\sim$ massless modes of V-twisted Dirac Operator: $\not\nabla_{X,V}\Psi = 0$
  - massless modes of $\not\nabla_{X,V} \stackrel{1:1}{\longleftrightarrow}$ V-valued cohomology groups
- Gauge group$(V) = G = SU(n)$, $n = 3, 4, 5$, gives H = Commutant(G, E_8):

$E_8 \to G \times H$	Breaking Pattern		
$SU(3) \times E_6$	248	$\to$	$(1,78) \oplus (3,27) \oplus (\bar{3},\overline{27}) \oplus (8,1)$
$SU(4) \times SO(10)$	248	$\to$	$(1,45) \oplus (4,16) \oplus (\bar{4},\overline{16}) \oplus (6,10) \oplus (15,1)$
$SU(5) \times SU(5)$	248	$\to$	$(1,24) \oplus (5,\overline{10}) \oplus (\bar{5},10) \oplus (10,5) \oplus (\overline{10},\bar{5}) \oplus (24,1)$

- Particle content

Decomposition	Cohomologies
$\mathrm{SU}(3) \times \mathrm{E}_6$	$n_{27} = h^1(V)$, $n_{\overline{27}} = h^1(V^*) = h^2(V)$, $n_1 = h^1(V \otimes V^*)$
$\mathrm{SU}(4) \times \mathrm{SO}(10)$	$n_{16} = h^1(V)$, $n_{\overline{16}} = h^2(V)$, $n_{10} = h^1(\wedge^2 V)$, $n_1 = h^1(V \otimes V^*)$
$\mathrm{SU}(5) \times \mathrm{SU}(5)$	$n_{10} = h^1(V^*)$, $n_{\overline{10}} = h^1(V)$, $n_5 = h^1(\wedge^2 V)$, $n_{\bar{5}} = h^1(\wedge^2 V^*)$, $n_1 = h^1(V \otimes V^*)$

- Further to SM: $H \xrightarrow{\text{Wilson Line}} SU(3) \times SU(2) \times U(1)$

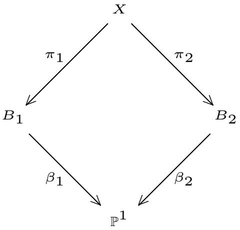
Ubi Materia, Ibi Geometria

- Issues in low-energy physics $\sim$ Precise questions in Alg Geo of (X, V)
  - Particle Content $\sim$ (tensor powers) V Equivariant Bundle Cohomology on X
  - LE SUSY $\sim$ Hermitian Yang-Mills connection $\sim$ Bundle Stability
  - Yukawa $\sim$ Trilinear (Yoneda) composition
  - Doublet-Triplet splitting $\sim$ representation of fundamental group of X
- e.g., for $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$ WL:

Cohomology	Representation	Multiplicity	Name
$[\alpha_1^2 \alpha_2 \otimes H^1(X,V)]^{inv}$	$(\mathbf{3},\mathbf{2})_{1,1}$	3	left-handed quark
$[\alpha_1^2 \otimes H^1(X,V)]^{inv}$	$(\mathbf{1},\mathbf{1})_{6,3}$	3	left-handed anti-lepton
$[\alpha_1^2 \alpha_2^2 \otimes H^1(X,V)]^{inv}$	$(\overline{\mathbf{3}},\mathbf{1})_{-4,-1}$	3	left-handed anti-up
$[\alpha_2^2 \otimes H^1(X,V)]^{inv}$	$(\overline{\mathbf{3}},\mathbf{1})_{2,-1}$	3	left-handed an ti-down
$[H^1(X,V)]^{inv}$	$(\mathbf{1},\mathbf{2})_{-3,-3}$	3	left-handed lepton
$[\alpha_1 \otimes H^1(X,V)]^{inv}$	$(\mathbf{1},\mathbf{1})_{0,3}$	3	left-handed anti-neutrino
$[\alpha_1 \otimes H^1(X,\wedge^2 V)]^{inv}$	$(\mathbf{1},\mathbf{2})_{3,0}$	1	up Higgs
$[\alpha_1^2 \otimes H^1(X,\wedge^2 V)]^{inv}$	$(\mathbf{1},\mathbf{2})_{-3,0}$	1	down Higgs

A Heterotic Standard Model

- [Braun-YHH-Ovrut-Pantev] (hep-th/0512177, 0601204)



- $X_0^{19,19}$ double-fibration over dP_9 $\quad \pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$
- V stable $SU(4)$ bundle: Generalised Serre Constrct
- Couple to $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson Line
- Matter $= \mathbb{Z}_3 \times \mathbb{Z}_3$-Equivariant cohomology on $X_0^{3,3}$

- Exact $SU(3) \times SU(2) \times U(1) \times U(1)_{B-L}$ spectrum:

  No exotics; no anti-generation; 1 pair of Higgs; RH Neutrino

- $SU(5) \to SU(3) \times SU(2) \times U(1)$ version [Bouchard-Cvetic-Donagi]

  same manifold

- $X_0^{19,19}$ is a CICY! Obvervatio Curiosa

Algorithmic Compactification

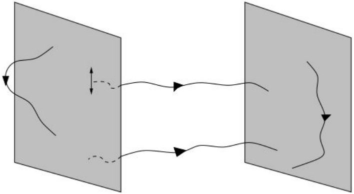
- Searching the MSSM, *Sui Generis?*
  - $\sim 10^7$ Spectral Cover bundles [Donagi, Friedman-Morgan-Witten, 1996-8] over elliptically fibered CY3 (2005-9), [Donagi-YHH-Ovrut-Pantev-Reinbacher, Gabella-YHH-Lukas,. . .]
  - $\sim 10^5$ (Monad) Bundles over all CICYs [Anderson-Gray-YHH-Lukas, 2007-9]
  - Monad Bundles over KS YHH-Kreuzer-Lee-Lukas 2010-11: ~ 200 in 10^5 3-gens
  - culminating in .. Stable Sum of Line Bundles over CICYs (Oxford-Penn-Virginia 2012-) Anderson-Gray-Lukas-Ovrut-Palti ~ 200 in 10^{10} MSSM
- meanwhile . . . LANDSCAPE grew with D-branes Polchinski 1995, M-Theory/G_2 Witten, 1995, F-Theory/4-folds Katz-Morrison-Vafa, 1996, AdS/CFT Maldacena 1998, Flux-compactification Kachru-Kallosh-Linde-Trivedi, 2003, . . .

Part II

Branes and the Non-Compact Calabi-Yau Landscape

Recall: D-branes in Type IIB

- D-branes Dirichlet Boundary conditions for open strings;
- D-brane world-volumes: Dp has $p+1$-D w.v.



$D1, D3, \ldots, D9$ of dimensions $1+1, \ldots, 9+1$;

DYNAMICAL: Carry charges $(2, 4, \ldots, 10$ forms) $\int_{Dp} Q^{(p+1)}$

- i.e., Open strings carry charges (Chan-Paton factors) $\Rightarrow$

  D-branes = Supports of Sheafs (strictly: D-brane = object in $D^b(Coh)$)

Another $10 = 4 + 6$

- important property: GAUGE ENHANCEMENT
  - i.e., *world-volume sees a* $U(1)$*-bundle*
  - Bringing together (stack) n parallel D-branes $U(1)^n \to U(n)$
- SUMMARY Type IIB: 10D, Closed Strings, Open Strings/Dp-Branes, p odd
- $\mathbb{R}^{1,9} \simeq \mathbb{R}^{1,3}$(world-volume of D3) $\times X^6$(transverse non-compact CY3)
- SIMPLEST CASE: transverse CY3 $= \mathbb{C}^3$
  - Original Maldacena's AdS/CFT (1997): $\mathcal{N} = 4$ U(n) SYM on 4D probe w.v.
  - Gauge Fields A^μ: $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$
  - Matter Fields $\mathcal{R} = \mathbf{4}, \mathbf{6}$: Adjoint (Weyl) fermions $\Psi^{\mathbf{4}}_{IJ}$: $\mathbf{4} \otimes \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$

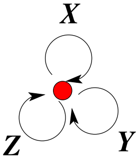
    Bosons $\Phi^{\mathbf{6}}_{IJ}$: $\mathbf{6} \otimes \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$

A Geometer's AdS/CFT

- Rep. Variety(Quiver) $\sim$ VMS(SUSY QFT) $\sim$ affine/singular variety

  e.g $\mathcal{N}=1$ Quiver variety = vacuum of F- & D-flatness = non-compact CY3

- $\mathcal{N}=4$ $U(N)$ Yang-Mills
  - 3 adjoint fields X, Y, Z with superpotential $W = \text{Tr}(XYZ - XZY)$



  - N D3-branes (w.v. is $\mathcal{N}=4$ in $\mathbb{R}^{3,1}$) $\perp \mathbb{R}^6$

    $\simeq \mathbb{C}^3$ = Vacuum Moduli Space
  - VMS $\simeq$ affine non-compact CY3 by construction

  - QUIVER = Finite graph (label = rk(gauge factor)) + relations from W
    - Matter Content: Nodes + arrows
    - Relations (F-Terms): $D_i W = 0 \rightsquigarrow [X,Y]=[Y,Z]=[X,Z]=0$
  - Here $\mathbb{C}^3$ is real cone over S^5 (simplest Sasaki-Einstein 5-manifold), others?

Orbifolds (V-manifolds)

- Orbifolds: next best thing to $\mathbb{C}^3$ (Satake 60's);
- Transverse CY3 $\simeq \mathbb{C}^3/\{\Gamma \subset SU(k)\}$ that admit crepant resolution, i.e., resolve to Calabi-Yau; Γ discrete finite subgroup of holonomy $SU(k)$; $k = 2, 3$
- Γ-Projection: $\gamma A^\mu \gamma^{-1} = A^\mu$ and $\Psi_{IJ} = R(\gamma)\gamma\Psi_{IJ}\gamma^{-1}$; i.e.,
  - Gauge Group $U(n) \Rightarrow \prod_i U(N_i)$
  - Matter fields decompose as

$$\begin{aligned} (\mathcal{R} \otimes \hom(\mathbb{C}^n, \mathbb{C}^n))^\Gamma &= \bigoplus_{i,j} \mathcal{R} \otimes \left(\mathbb{C}^{N_i} \otimes \mathbb{C}^{N_j *} \otimes \mathbf{r_i} \otimes \mathbf{r_j^*}\right)^\Gamma \\ &= \bigoplus_{i,j} a_{ij}^{\mathcal{R}} \left(\mathbb{C}^{N_i} \otimes \mathbb{C}^{N_j *}\right), \end{aligned}$$

where $\boxed{\mathcal{R} \otimes \mathbf{r}_i = \bigoplus_j a_{ij}^{\mathcal{R}} \mathbf{r}_j}$

  - $a_{ij}^{\mathbf{4}}$ bi-fundamental fermions: $(N_i, \bar{N}_j)$ of $SU(N_i) \times SU(N_j)$
  - $a_{ij}^{\mathbf{6}}$ bi-fundamental bosons: $(N_i, \bar{N}_j)$ of $SU(N_i) \times SU(N_j)$

Quivers

-

	Parent	$\xrightarrow{\Gamma}$	Orbifold Theory
SUSY	$\mathcal{N}=4$	$\leadsto$	$\mathcal{N}=2$, for $\Gamma \subset SU(2)$ $\mathcal{N}=1$, for $\Gamma \subset SU(3)$ $\mathcal{N}=0$, for $\Gamma \subset \{SU(4) \simeq SO(6)\}$
Gauge Group	$U(n)$	$\leadsto$	$\prod\limits_i U(N_i)$, $\quad \sum\limits_i N_i \dim \mathbf{r}_i = n$
Fermion	$\Psi^{\mathbf{4}}_{IJ}$	$\leadsto$	$\Psi^{ij}_{f_{ij}}$
Boson	$\Phi^{\mathbf{6}}_{IJ}$	$\leadsto$	$\Phi^{ij}_{f_{ij}}$ $\quad \mathcal{R} \otimes \mathbf{r}_i = \bigoplus\limits_j a^{\mathcal{R}}_{ij} \mathbf{r}_j$

$I, J = 1, \ldots, n;\ f_{ij} = 1, \ldots, a^{\mathcal{R}=\mathbf{4},\mathbf{6}}_{ij}$

- In physics: Douglas & Moore (9603167), $\mathbb{C}^2/\mathbb{Z}_n$; Johnson & Meyers (9610140) Formalised in Lawrence, Nekrasov & Vafa, (9803015);

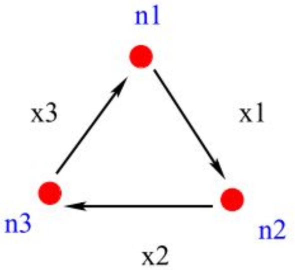
Quivers: Finite Graphs with Representation

- A Graphical way to represent this data
  - Node $i \sim$ gauge factor $U(N_i)$
  - Arrow $i \to j \sim$ bi-fundamental $(N_i, \bar{N}_j)$

- e.g.

  Adjacency Matrix

$$A_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



- Gabriel: 1970s: $x_1 \in \mathrm{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2})$, etc.

McKay Correspondence

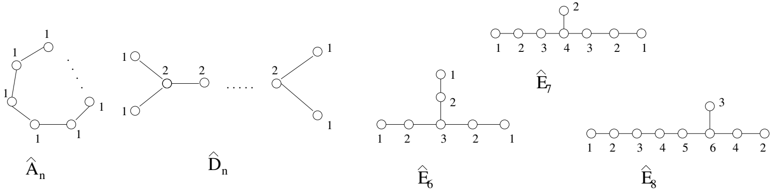
- Take the $\mathbb{C}^2/(\Gamma \subset SU(2)) \times \mathbb{C}$ case: Discrete Finite Subgroups of $SU(2)$
- F. Klein (1884) (double covers of those of $SO(3)$, i.e., symmetry groups of the Platonic solids)

Group	Name	Order
$A_n \simeq \mathbb{Z}_{n+1}$	Cyclic	$n+1$
D_n	Binary Dihedral	$2n$
E_6	Binary Tetrahedral	24
E_7	Binary Octahedral (Cube)	48
E_8	Binary Icosahedral (Dodecadedron)	120

- McKay (1980) Take the Clebsch-Gordan decomposition for $\mathcal{R}$ = fundamental **2** representation of $SU(2)$

ADE-ology

- $\mathbf{2} \otimes \mathbf{r}_i = \bigoplus_j a_{ij}^{\mathbf{2}} \mathbf{r}_j$ and treat $a_{ij}^{\mathbf{2}}$ as adjacency matrix
- McKay Quivers (rmk: Cartan matrix symmetric $\rightsquigarrow$ graph unoriented)
- QUIVERS = DYNKIN DIAG. OF CORRESPONDING AFFINE LIE ALGEBRA!!



Geometrical McKay

- Geometrically: González-Springberg & Verdier (1981)

  Crepant Resolution $K3 \to \mathbb{C}^2/\Gamma$

$$
\begin{aligned}
A_n : &\quad xy + z^n = 0 \\
D_n : &\quad x^2 + y^2 z + z^{n-1} = 0 \\
E_6 : &\quad x^2 + y^3 + z^4 = 0 \\
E_7 : &\quad x^2 + y^3 + yz^3 = 0 \\
E_8 : &\quad x^2 + y^3 + z^5 = 0
\end{aligned}
$$

- Intersection matrix of -2 exceptional curves in the blowup $\rightsquigarrow$ Quiver
- Bridgeland-King-Reid (1999) Use Fourier-Mukai: McKay as an auto-equivalence in $\mathcal{D}^b(\mathrm{coh}(\widetilde{X/G})) = \mathcal{D}^b(\mathrm{coh}^G(X))$

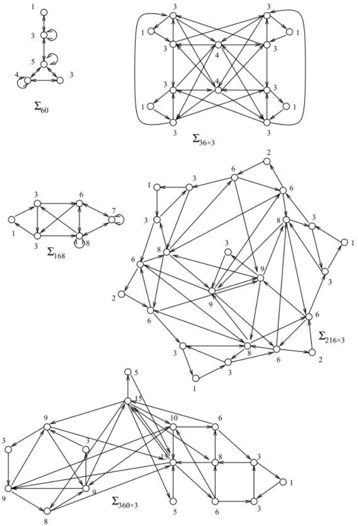
CY3 case: $\mathbb{C}^3/(\Gamma \subset SU(3))$

- *McKay Quiver* $\Rightarrow \mathcal{N} = 2$ *SUSY gauge theory on 4D world-volume*
- $\mathcal{N} = 1$ SUSY: Need discrete finite groups $\Gamma \subset SU(3)$
- Classification: Blichfeldt (1917)

Infinite Series	$\Delta(3n^2), \Delta(6n^2)$
Exceptionals	$\Sigma_{36\times 3}, \Sigma_{60\times 3}, \Sigma_{168\times 3}, \Sigma_{216\times 3}, \Sigma_{360\times 3}$

- Gives chiral $\mathcal{N} = 1$ gauge theories in 4D wv of D3-probe
- most phenomenologically interesting
- Hanany & YHH hep-th/9811183
- Rmk: Crepant Resolutions to CY3 and Generalised McKay (Reid, Ito et al.) not as well established

SU(3) quivers and $\mathcal{N} = 1$ gauge theories



$\Gamma \subset SU(3)$	Gauge Group
$\widehat{A_n} \simeq \mathbb{Z}_{n+1}$	(1^{n+1})
$\mathbb{Z}_k \times \mathbb{Z}_{k'}$	$(1^{kk'})*$
$\widehat{D_n}$	$(1^4, 2^{n-3})$
$\widehat{E_6} \simeq \mathcal{T}$	$(1^3, 2^3, 3)$
$\widehat{E_7} \simeq \mathcal{O}$	$(1^2, 2^2, 3^2, 4)$
$\widehat{E_8} \simeq \mathcal{I}$	$(1, 2^2, 3^2, 4^2, 5, 6)$
$E_6 \simeq T$	$(1^3, 3)$
$E_7 \simeq O$	$(1^2, 2, 3^2)$
$E_8 \simeq I$	$(1, 3^2, 4, 5)$
$\Delta_{3n^2} (n = 0 \bmod 3)$	$(1^9, 3^{\frac{n^2}{3}-1})*$
$\Delta_{3n^2} (n \neq 0 \bmod 3)$	$(1^3, 3^{\frac{n^2-1}{3}})*$
$\Delta_{6n^2} (n \neq 0 \bmod 3)$	$(1^2, 2, 3^{2(n-1)}, 6^{\frac{n^2-3n+2}{6}})*$
Σ_{168}	$(1, 3^2, 6, 7, 8)*$
Σ_{216}	$(1^3, 2^3, 3, 8^3)$
$\Sigma_{36\times3}$	$(1^4, 3^8, 4^2)*$
$\Sigma_{216\times3}$	$(1^3, 2^3, 3^7, 6^6, 8^3, 9^2)*$
$\Sigma_{360\times3}$	$(1, 3^4, 5^2, 6^2, 8^2, 9^3, 10, 15^2)*$

DICTIONARY: Quivers & Gauge Theory

$$S = \int d^4x \left[\int d^2\theta d^2\bar{\theta}\ \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2}\int d^2\theta\ \mathrm{Tr}\,\mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta\ W(\Phi) + \text{c.c.}\right)\right]$$

W = superpotential $\qquad V(\phi_i, \bar{\phi}_i) = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + \frac{g^2}{4}(\sum_i q_i |\phi_i|^2)^2$

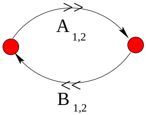
- Encode into QUIVER (rep of finite labelled graph with relations):

k nodes, dim vec $(N_1, \ldots, N_k)$	$\prod_{j=1}^k U(N_j)$ gauge group
Arrow $i \to j$	bi-fund X_{ij} field $(\square, \bar{\square})$ of $U(N_i) \times U(N_j)$
Loop $i \to i$	adjoint ϕ_i field of $U(N_i)$
Cycles	Gauge Invariant Operator
2-cycles	Mass-terms
$W = \sum c_i$ cycles$_i$	Superpotenital
Relations	Jacobian of $W(\phi_i, X_{ij})$

- VACUUM $\sim V(\phi_i, \bar{\phi}_i) = 0 \Rightarrow \begin{cases} \frac{\partial W}{\partial \phi_i, X_i} = 0 & \text{F-TERMS} \\ \sum_i q_i |\phi_i|^2 + q_k |X_k| = 0 & \text{D-TERMS} \end{cases}$

Another Famous Example: Conifold

- $SU(N) \times SU(N)$ gauge theory with 4 bi-fundamental fields



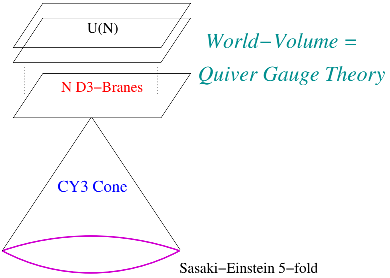
QUIVER

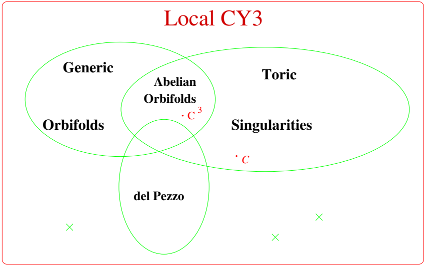
	$SU(N)$	$SU(N)$
$A_{i=1,2}$	$\square$	$\overline{\square}$
$B_{j=1,2}$	$\overline{\square}$	$\square$

$W = \text{Tr}(\epsilon_{il}\epsilon_{jk} A_i B_j A_l B_k)$

- D3-branes transverse to the conifold singularity $= (\{uv = wz\} \subset \mathbb{C}^4) =$ VMS (Klebanov-Witten 1999] $\mathcal{N} = 1$ "conifold" Theory)
- # gauge factors $= N_g = 2$; # fields $= N_f = 4$; # terms in $W = N_w = 2$
- Observatio Curiosa: $N_g - N_f + N_w = 0$, as with $\mathbb{C}^3$, true for almost all known cases in AdS_5/CFT_4

- 2 decade programme of the School of A. Hanany:



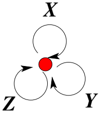


- Orbifolds: $\mathbb{C}^3/(\Gamma \subset SU(3))$ Generalized McKay Correspondence (Hanany-YHH, 98); Fano (del Pezzo): $dP_{0,\ldots,8}$ (w/ Hanany,Feng, Franco, et al. 98 - 00); LARGEST FAMILY by far Toric: e.g., conifold, $Y^{p,q}$, $L^{p,q}$ ...

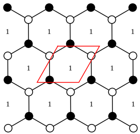
$\mathcal{M}$ Toric CY3 $\longleftrightarrow$ Bipartite Graph on T^2

Feng, Franco, Hanany, YHH, Kennaway, Martelli, Mekareeya, Seong, Sparks, Vafa, Vegh, Yamazaki, Zaffaroni . . .

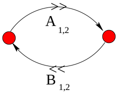
- $N_g - N_f + N_w = 0$ is Euler relation for a tiling of torus
- Jac(W) = binomial ideal (toric): bipartite Notation for Toric Cones



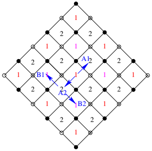
$\rightarrow$



$$W = \text{Tr}(XYZ - XZY)$$



$\rightarrow$



$$W = \text{Tr}(\epsilon_{il}\epsilon_{jk} A_i B_j A_l B_k)$$

Toric CY3, Mirror Symmetry & Bipartite Tilings

- Mirror Symmetry [Strominger-Yau-Zaslow; Hori-Vafa]

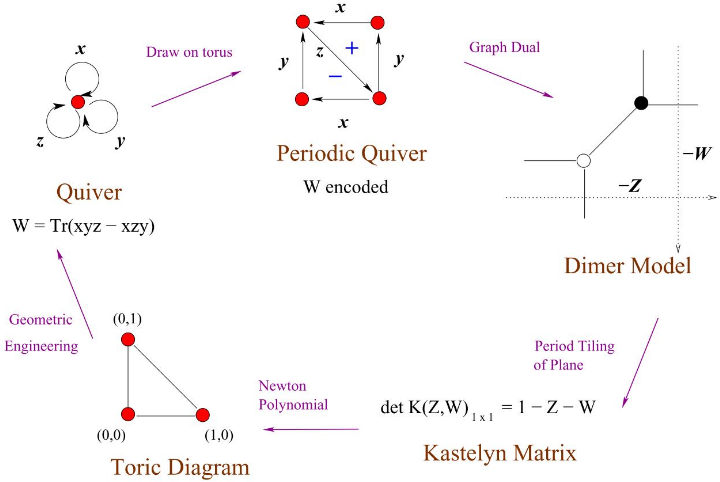
  D3-brane on CY3 $\rightsquigarrow$ D6-branes wrapping 3-cycles in mirror CY3
- [Feng-Kennaway-YHH-Vafa] torus T^2 lives in T^3 of mirror symmetry;

  Tropical Geometry
- THEOREM: [R. Böckland, N. Broomhead, A. Craw, A. King, K. Ueda ...]

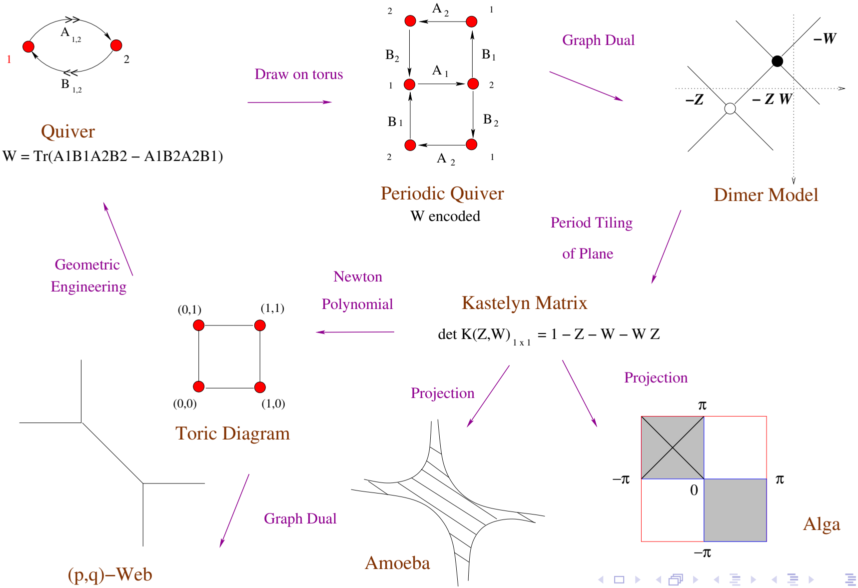
  The (coherent component of) VMS as representation variety of a quiver is an affine (non-compact, possibly singular) toric Calabi-Yau variety of complex dimension 3 $\Leftrightarrow$ the quiver + superpotential is graph dual to a bipartite graph drawn on T^2
- Rmk: Each $\Rightarrow$ SCFT in 3+1-d

SUMMARY: $\mathbb{C}^3$, Hexagonal Tilings, SYM

$\mathcal{N} = 1$ SYM $=$ D3-branes transverse to $\mathbb{C}^3 = \mathcal{C}(S^5)$ $=$ hexagonal bipartite tiling



SUMMARY: Conifold and Square Tilings



A QFT Duality & a Quiver Transformation

- Seiberg (1994): dual quantum field theories, in particular same VMS
  2 theories: Direct Electric theory: N_c with N_f flavours; Dual Magnetic theory: $N_f - N_c$ (take $\frac{3}{2}N_c \le N_f \le 3N_c$) with N_f flavours

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
Q	$\square$	$\square$	1
Q'	$\overline{\square}$	1	$\overline{\square}$

$W = 0$

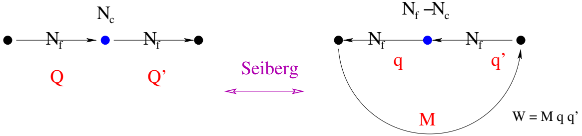
	$SU(N_f - N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
q	$\square$	$\overline{\square}$	1
q'	$\overline{\square}$	1	$\square$
M	1	$\square$	$\overline{\square}$

$W = Mqq'$

- Feng-Hanany-YHH (2000) using Hanany-Witten (1996)

  [cf. Cachazo-Intriligator-Katz-Vafa, 2001];

- Fomin-Zelevinsky [2000]: cluster mutation, completely independently!



A Quiver Duality from Seiberg Duality

We have quiver labeled by $(N_c)_i$ and arrows a_{ij}:

1. Pick dualisation node i_0 with N_c, an define:

$$
\begin{aligned}
I_{in} &:= \text{nodes having arrows going into } i_0 \\
I_{out} &:= \text{nodes having arrow coming from } i_0 \\
I_{no} &:= \text{nodes unconnected with } i_0
\end{aligned}
$$

2. $N_c \to N_f - N_c$ (where $N_f = \sum\limits_{i \in I_{in}} a_{i,i_0} = \sum\limits_{i \in I_{out}} a_{i_0,i}$)
3. Reverse arrows going in or out of i_0, leave I_{no}, and change affected nodes:

$$
a_{ij}^{dual} = \begin{cases} a_{ji} & \text{if either} \quad i, j = i_0 \\ a_{ij} - a_{i_0 i} a_{j i_0} & \text{if both} \quad A \in I_{out}, B \in I_{in} \end{cases}
$$

If negative, take it to mean $-a^{dual}$ arrows from j to i.

4. Generate W term

Grothendieck's Dessin d'Enfant *In Piam Memoriam*

- Belyǐ Map: Σ smooth compact Riemann surface, rational map $\beta : \Sigma \longrightarrow \mathbb{P}^1$ ramified only at $(0, 1, \infty)$
- **Theorem [Belyǐ]:** β exists $\Leftrightarrow$ Σ can be defined over $\overline{\mathbb{Q}}$; (β, Σ) Belyǐ Pair
- A Bipartite graph on Σ
  - label each $\beta^{-1}(0)$ black with valency = ramification index;
  - likewise $\beta^{-1}(1)$ white;
  - then $\beta^{-1}(\infty)$ fixed to live one per face
- Dessin d'Enfant $= \beta^{-1}([0,1] \in \mathbb{P}^1)$
- Ramification data / Passport: $\left\{ \begin{array}{l} r_0(1), r_0(2), \ldots, r_0(B) \\ r_1(1), r_1(2), \ldots, r_1(W) \\ r_\infty(1), r_\infty(2), \ldots, r_\infty(I) \end{array} \right\}$

Permutation Triples

- equivalent description Permutation Triple: Let there be d edges in the bipartite graph and consider symmetric group S_d, define in cycle-notation

$$\sigma_B = (\ldots)_{r_0(1)}(\ldots)_{r_0(2)}\cdots(\ldots)_{r_0(B)} \qquad \sigma_B\sigma_W\sigma_\infty = \mathbb{I}$$
$$\sigma_W = (\ldots)_{r_1(1)}(\ldots)_{r_1(2)}\cdots(\ldots)_{r_1(W)}$$

encodes how the sheets are permuted at the ramification points;

- Example: $\sigma_B = \sigma_W = \sigma_\infty = (123)$

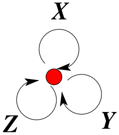
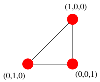
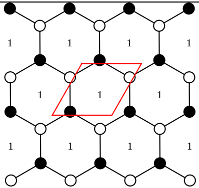
$\mathbb{T}^2 : y^2 = x^3 + 1$	$\xrightarrow{\beta=\frac{1}{2}(1+y)}$	$\mathbb{P}^1$	Local Coordinates on $\mathbb{T}^2$	Ramification Index of β
$(0, -1)$	$\stackrel{\beta}{\mapsto}$	0	$(x, y) \sim (\epsilon, -1 - \frac{1}{2}\epsilon^3)$	3
$(0, 1)$	$\stackrel{\beta}{\mapsto}$	1	$(x, y) \sim (\epsilon, 1 + \frac{1}{2}\epsilon^3)$	3
(∞, ∞)	$\stackrel{\beta}{\mapsto}$	∞	$(x, y) \sim (\epsilon^{-2}, \epsilon^{-3})$	3

Gauge Theories and Dessins

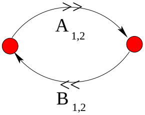
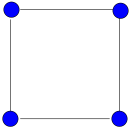
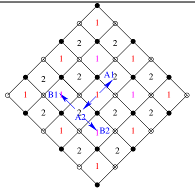
- Toric CY3 Quiver $\rightsquigarrow$ bipartite tiling of T^2 $\rightsquigarrow$ Belyǐ pair

$$\big(\text{Elliptic Curve } E, \qquad \beta : E \longrightarrow \mathbb{P}^1\big)$$

- Our most familiar example of $\mathcal{N} = 4$ super-Yang-Mills:

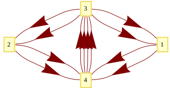
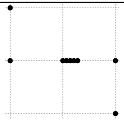
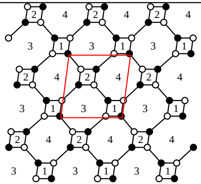
Theory	Toric Diag	Belyǐ Pair	Dessin on T^2 (dimer)
 $W = \text{Tr}(X[Y,Z])$	 $\mathcal{M} \simeq \mathbb{C}^3$	$y^2 = x^3 + 1$ $\beta(x,y) = \frac{y+1}{2}$	

- Klebanov-Witten's Conifold Theory

Theory	Toric Diag
 $W = \mathrm{Tr}(\epsilon_{il}\epsilon_{jk}A_iB_jA_lB_k)$	 $\mathcal{M} \simeq \{uv - wz = 0\} \subset \mathbb{C}^4$
Belyǐ Pair	**Dessin on T^2 (dimer)**
$y^2 = x(x-1)(x-\frac{1}{2})$ $\beta(x,y) = \frac{x^2}{2x-1}$	

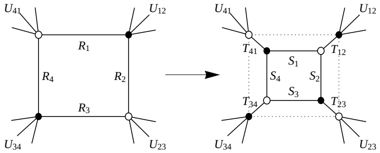
Plethora of Non-Trivial Examples

e.g., Cone over $F_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (zeroth Hirzebruch surface);

Theory	Toric Diag
 $W = X^1_{14}X^4_{43}X^2_{31} + X^2_{14}X^2_{43}X^1_{31} + X^1_{24}X^1_{43}X^1_{32} + X^2_{24}X^3_{43}X^2_{32} - X^1_{43}X^1_{31}X^1_{14} - X^2_{43}X^2_{32}X^1_{24} - X^3_{43}X^2_{31}X^2_{14} - X^4_{43}X^1_{32}X^2_{24}$	 $\mathcal{M} \simeq$Hirzebruch 0
Belyĭ Pair	**Dessin on T^2 (dimer)**
$y^2 = x - x^3$ $\beta(x,y) = \frac{i\left(x^2 - \sqrt[3]{-1}\right)^3}{3\sqrt{3}x^2(x^2-1)}$	

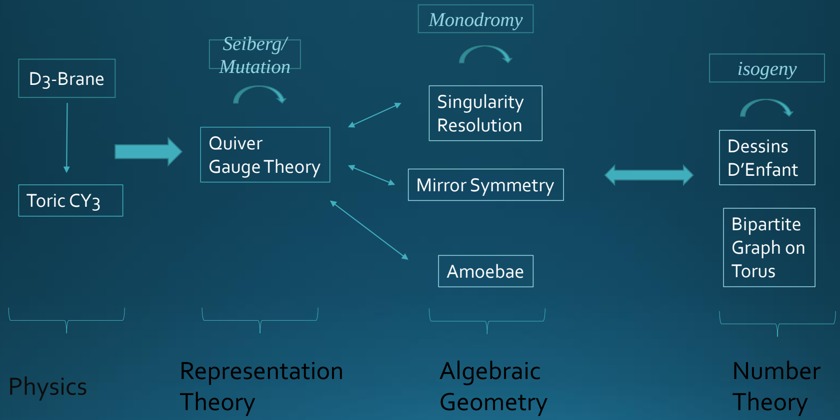
Rigidity & Transcendence Degree

- Dessins are rigid: in particular elliptic curve has fixed τ
- In gauge theory:
    - R-charge $\sim$ length(edges), choose isoradial embedding (all nodes are on circles of equal radius); then fix by a-max $=$ volume Z-min of Sasaki-Einstein (Intriligator-Wecht, Martelli-Sparks-Yau); Futaki-Donaldson Inv.
    - R-charges and normalized volume of dual geometry are *algebraic numbers*
- Seiberg Duality/Cluster Mutation $=$ so-called "Urban Renewal"



- $j(\tau)$ of isoradial dimer invariant:
- transcendence degree $/\mathbb{Q}$ of R-charges invariant

SUMMARY


D3-Brane
Toric CY3
Seiberg/ Mutation
Quiver Gauge Theory
Monodromy
Singularity Resolution
Mirror Symmetry
Amoebae
isogeny
Dessins D'Enfant
Bipartite Graph on Torus
Physics
Representation Theory
Algebraic Geometry
Number Theory

Open Problems

- relation amongst the 3 complex structures?

Physics	Geometry	Number Theory
τ(a-max/Vol-min)	τ(mirror)	τ(dessin)

- Define $\mathcal{D}^g_{\geq 3} := \{$dessins of valency ≥ 3 on $\Sigma_g\}$ then Observation:

$$\Psi : \mathcal{D}^g_{\geq 3} \twoheadrightarrow \left\{\text{affine toric } CY^{2g+1}\right\}$$

  Ψ surjection (by having CY^{2g+1} as representation variety of dual quiver)

- Conjecture: $\Psi^{-1}(\mathcal{M})$ in orbits of cluster mutation/Seiberg/urbal renewal
- Question: Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{D}^g_{\geq 3}$ (faithful for g $=$ 0,1), what is action on $\left\{\text{affine toric } CY^{2g+1}\right\}$ and on quiver gauge theory?

Part III

WWJD: What Would JPython/AI Do?

YHH, 1706.02714, PLB 774, 2017

SUMMARY: Algorithms and Datasets in String Theory

- Growing databases and computational algorithms motivated by string theory
- Archetypical Problems
  - Classify configurations (typically integer matrices: polyotope, adjacency, ...)
  - Compute geometrical quantity algorithmically
    - toric $\rightsquigarrow$ combinatorics;
    - quotient singularities $\rightsquigarrow$ rep. finite groups;
    - generically $\rightsquigarrow$ ideals in polynomial rings;
    - Numerical geometry (homotopy continuation);
    - Cohomolgy (spectral sequences, Adjunction, Euler sequences)
- Typical Problem in String Theory/Algebraic Geometry:

$$\underset{\boxed{\text{integer tensor}}}{INPUT} \longrightarrow \underset{\boxed{\text{integer}}}{OUTPUT}$$

Where we stand . . .

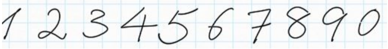
The Good — Last 10-15 years: several international groups have bitten the bullet Oxford, London, Vienna, Blacksburg, Boston, Johannesburg, Munich, . . . computed many geometrical/physical quantities and compiled them into various databases **Landscape Data** ($10^{9\sim10}$ entries typically)

The Bad — Generic computation HARD: dual cone algorithm (exponential), triangulation (exponential), Gröbner basis (double-exponential) . . . e.g., how to construct stable bundles over the $\gg 473$ million KS CY3? Sifting through for MSSM not possible . . .

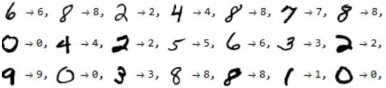
The ??? — **Borrow new techniques from "Big Data" revolution**

A Prototypical Question

- Hand-writing Recognition, e.g., my 0 to 9 is different from yours:



- How to set up a bijection that takes these to $\{1, 2, \ldots, 9, 0\}$? Find a clever Morse function? Compute persistent homology? Find topological invariants? ALL are inefficient and too sensitive to variation.
- What does your iPhone/tablet do? What does Google do?
    - Take large sample, take a few hundred thousand (e.g. NIST database)



. . .

    - Machine-Learn: (1) Data Acquisition; (2) Setup Neural Network (NN); (3) Train NN. generically, if the NN is sufficiently complex, called Deep Learning

A Single Neuron: The Perceptron

- began in 1957 (!!) in early AI experiments (using CdS photo-cells)
- DEF: Imitates a neuron: activates upon certain inputs, so define
  - Activation Function $f(z_i)$ for input tensor z_i for some multi-index i;
  - consider: $f(w_i z_i + b)$ with w_i weights and b bias/off-set;
  - typically, $f(z)$ is sigmoid, Tanh, etc.
- Given training data: $D = \{(x_i^{(j)}, d^{(j)}\}$ with input x_i and **known output** $d^{(j)}$, minimize

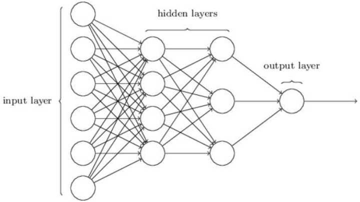
$$SD = \sum_j \left(f(\sum_i w_i x_i^{(j)} + b) - d^{(j)} \right)^2$$

to find optimal w_i and $b \rightsquigarrow$ "learning"

- Essentially (non-linear) regression

The Neural Network: network of neurons $\rightsquigarrow$ the "brain"

- DEF: a connected graph, each node is a perceptron (Beta-version implemented on Mathematica 11.1 +)
    1. adjustable weights/bias;
    2. distinguished nodes: 1 set for input and 1 for output;
    3. iterated training rounds.



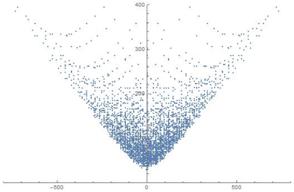
Simple case: forward directed only, called multilayer perceptron

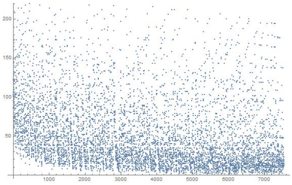
- use the simple MLP: e.g., Sigmoid $\to$ Linear $\to$ Tanh $\to$ Summation
- Essentially how brain learns complex tasks; **apply to our Landscape Data**

Hypersurfaces in $W\mathbb{P}^4$: Warmup I

Oftentimes, questions in pheno are **qualitative**, e.g.,

- large # complex structure how many have, say, $h^{2,1} > 50$?
  - [Candelas-Lynker-Schimmrigk] Landau-Ginzburg methods: many hours; using Euler sequence/Adjunction: many more hours

(a) 

(b) 

(a) Mirror plot of $(\chi, h^{1,1} + h^{2,1})$ (b) Distribution of $h^{2,1}$

  - With the MLP NN, 500 training rounds, under 1 min, learns $h^{2,1} > 50$ to 97% Cosine distance $D_C = 0.998$, Matthews $\phi = 0.84$.
  - consistency check (testing full set): cool and re-assuring but not useful

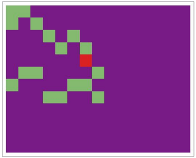
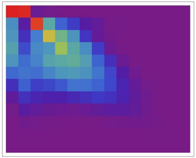
Hypersurfaces in $W\mathbb{P}^4$: Warmup II

- What if the data is not complete? Very often the case when computation powers are not yet capable (e.g., all triang for KS dataset: don't even know how many CY3 hypersurfaces in the 473 million toric varieties)
- Standard method: take partial training and validation data, s.t., $D = T \sqcup V$
  - train NN with random 2000/7555 inputs ($\sim 1/4$ only)
  - use the trained NN to predict value for the remaining UNSEEN 7555 - 2000
  - Get $\sim 91.8\%$ precision, $d_C = 0.91$, $\phi = 0.84$ in less than 20 sec on regular laptop! Learning Curves
- Another Question: How many have χ divisible by 3? (useful for # generations after Wilson line)

  2000 samples ~ 1 min: 80% precision, $d_C = 0.91$ when predicting 7555-2000
- **Endless possibilities of mathematical/physical queries. . .**

CICYs: a Colourful Example

- An image = a matrix (pixels) with entries denoting shade/colour; NN really good at images (e.g. hand-writing) [RMK: not using a convolutional NN here]
- CICY is a (padded) 12×15 matrix with 6 colours $\rightsquigarrow$ CICY is an image

(a)  (b) 

(a) typical CICY;
(b) average CICY

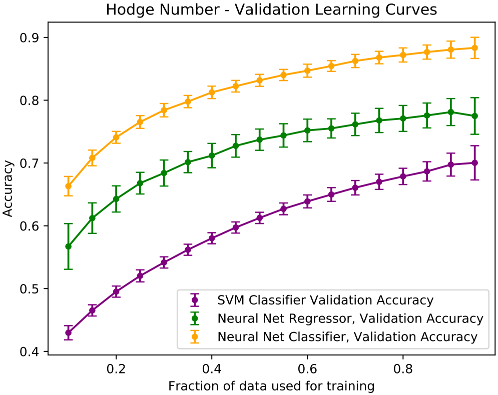
- Input more sophisticated, so greater accuracy expected: e.g. in learning large number of Kahler parametres $h^{1,1} > 5$:
  learns 4000 samples ($< 50\%$) in ~ 5 min; validate against 7890-4000: 97% accuracy, $d_C = 0.98$, $\phi = 0.87$.

Learning Curves

CICYs: Detailed Analysis

Kieran Bull [Oxford] [Bull-YHH-Jejjala-Mishra: arXiv:1806.03121]

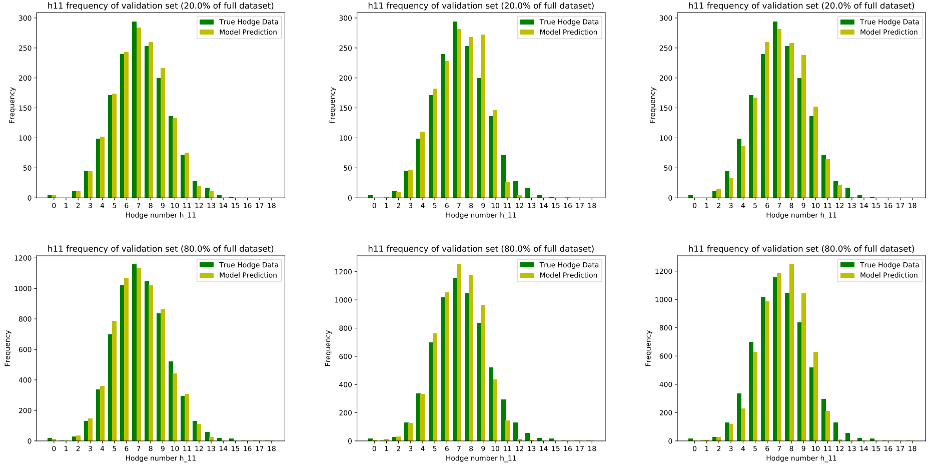
- **TensorFlow** Python's implementation of NNs and DL
- Compare NNs with Decision Trees, Support Vector Machines, etc



Can one learn the FULL information on Hodge numbers? $h^{1,1} \in [0, 19]$ so can set up 20-channel NN classifer, regressor, as well as SVM

CICYs: Comparative Studies

$h^{1,1}$ for NN, Regressor, SVM at 20 and 80% training Sky's the Limit



Remarks and Sanity Checks

- Why does it work?
  - Short answer in the data-science community: nobody knows!!
  - Theorems still need to be proven about convergence, measure, etc., esp. for a large number of neurons; even a few neurons has many parametres
  - At the most basic level: problems in algebraic geometry boil down to finding kernels of integer matrices
  - NOT over-fitting training data $\cap$ validation data $= \{\}$
- A Reprobate: Try to predict the next prime; has to fail, otherwise crazy
  - Train our NN: gets a miserable 0.1% accuracy even on learning, forget about predicting, great! Better off just fitting $n \log(n)$ using PNT
  - expect other things like digits of π to utterly fail

Summary and Outlook

PHYSICS

- The string landscape now solidly resides in the age of Big Data
- Use Neural Networks as
  1. Classifier deep-learn and categorize landscape data
  2. Predictor estimate results beyond computational power

MATHS

- somehow bypassing the expensive steps of long sequence-chasing, Gröbner bases, dual cones/combinatorics and getting the right answer. **how is AI doing maths more efficiently without knowing any maths?**
- problems in geometry, combinatorics, etc, good; number theory, not so good.

- many species of animals are capable of extremely sophisticated tasks (e.g., chimps with herbal medicine); we are such a species when confronted with the landscape; we can (deep-)learn by trial-error before we tackle the fundamental question of why in the future ...
- Try your favourite problem and see
- Boris Zilber [Merton Professor of Logic, Oxford]: "you've managed syntax without semantics..."

Sophia (Hanson Robotics, HK)

First non-human citizen (2017, Saudi Arabia)

First non-human with UN title (2017)

Thank you

...

大哉大哉，宇宙之謎。美哉美哉，真理之源。

時空量化，智者無何。管測大塊，學也洋洋。

丘成桐先生： 時空頌

Infinite, infinite the secrets of the universe.

Inexhaustible, lovely in every detail.

Measure time, measure space no one can do it.

Watched through a straw what's to be learned has no end.

Prof. Shing-Tung Yau, 2002

Some Rudiments & Nomenclature

A sequence of specializations:

- M Riemannian: positive-definite symmetric metric
- M Complex Riemannian: have (p,q)-forms with p-holomorphic and q-antiholomorphic indices: $d = \partial + \bar{\partial}$ (with $\partial^2 = \bar{\partial}^2 = \{\partial, \bar{\partial}\} = 0$)
- M Hermitian: complex Riemannian and can tranform $g_{mn} = g_{\bar{m}\bar{n}} = 0$
- M Kähler: Hermitian with Kähler form $\omega := i g_{m\bar{n}} dz^m \wedge dz^{\bar{n}}$ such that $d\omega = 0$ ($\Rightarrow \partial_m g_{n\bar{p}} = \partial_n g_{m\bar{p}}$; $g_{m\bar{n}} = \partial\bar{\partial}K(z,\bar{z})$ for some scalar K)

Cohomology:

- On Riemannian M: can define Laplacian on p-forms (Hodge star $\star(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}) := \frac{\epsilon^{\mu_1 \cdots \mu_n}}{(n-p)!\sqrt{|g|}} g_{\mu_{p+1}\nu_{p+1}} \cdots g_{\mu_n \nu_n} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_n}$)

$$\Delta_p = dd^\dagger + d^\dagger d = (d + d^\dagger)^2, \qquad d^\dagger := (-1)^{np+n+1} \star d \star$$

Harmonic p-Form $\Delta_p A^p = 0 \overset{1:1}{\longleftrightarrow} H^p_{deRham}(X)$

- On Hermitian M: Dolbeault Cohomology $H^{p,q}_{\bar{\partial}}(X)$: cohomology on $\bar{\partial}$ (similarly ∂) and $\Delta_\partial := \partial\partial^\dagger + \partial^\dagger\partial$ and similarly $\Delta_{\bar{\partial}}$
- On Kähler M: $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$, Hodge decomposition:

$$H^i(M) \simeq \bigoplus_{p+q=i} H^{p,q}(M)$$

Back to Calabi-Yau

Covariant Constant Spinor

- Define $J_m^n = i\eta_+^\dagger \gamma_m^n \eta_+ = -i\eta_-^\dagger \gamma_m^n \eta_-$, check: $J_m^n J_n^p = -\delta_m^n$
- (X^6, J) is thus almost-complex
- But η covariant constant $\rightsquigarrow \nabla_m J_n^p = 0 \rightsquigarrow \nabla N_{mn}^p = 0$

  Nijenhuis tensor $N_{mn}^p := J_m^q \partial_{[q} J_{n]}^p - (m \leftrightarrow n)$
- (X^6, J) is thus complex ($J_m^n = i\delta_m^n, \ J_{\bar{m}}^{\bar{n}} = i\delta_{\bar{m}}^{\bar{n}}, \ J_{\bar{m}}^n = J_m^{\bar{n}} = 0$ for some local coordinates $(z, \bar{z})$; transition functions holomorphic)
- Define $J = \frac{1}{2} J_{mn} dx^m \wedge dx^n$ ($J_{mn} := J_m^k g_{kn}$) check:

  $dJ = (\partial + \bar{\partial})J = 0$
- (X^6, J) is thus Kähler
- summary $\boxed{X^6 \text{ is a Kähler manifold of } \dim_{\mathbb{C}} = 3, \text{ with } SU(3) \text{ holonomy}}$

Famous CICYs

- The Quintic $Q = [4|5]^{1,101}_{-200}$ (or simply $[5]$);
- Tian-Yau Manifold: $TY = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}^{14,23}_{-18}$
  - no CICY has $\chi = \pm 6$
  - TY has freely-acting $\mathbb{Z}_3 \rightsquigarrow (TY/\mathbb{Z}_3)^{6,9}_{-6}$;
  - central to early string pheno [Distler, Greene, Ross, et al.]
- Schön Manifold: $S = \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}^{19,19}_{0}$ has $\mathbb{Z}_3 \times \mathbb{Z}_3$ freely acting symmetry
  - explored more recently;
  - The quotient is $M^0_{3,3}$.

Back to CICYs

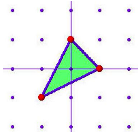
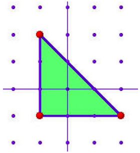
Reflexive Polytopes: Rudiments

- Convex Lattice Polytope Δ (use Δ_n to emphasize dim n)
  - DEF1 (Vertex Rep): Convex hull of set S of k lattice points $p_i \in \mathbb{Z}^n \subset \mathbb{R}^n$

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^{k} \alpha_i p_i | \alpha_i \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1 \right\}$$

  - DEF2 (Half-Plane Rep): intersection of integer inequalities $A \cdot \underline{x} \geq \underline{b}$
  - {extremal pts = vertices, edges, 2-faces, 3-faces, ..., (n-1)-faces = facets, Δ}
  - $n = 2$ polygons, $n = 3$ polyhedra, ...
- Polar Dual: $\Delta^\circ = \{\underline{v} \in \mathbb{R}^n \mid \underline{m} \cdot \underline{v} \geq -1 \ \ \forall \underline{m} \in \Delta\}$
- Reflexive Δ: if Δ° is also convex lattice polytope
  - in general, vertices of Δ° are rational, not integer
  - duality: $(\Delta^\circ)^\circ = \Delta$
  - if further $\Delta = \Delta^\circ$, self-dual/self-reflexive

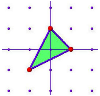
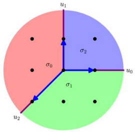
Reflexive Polytope: example

Δ_2		Vertices : $(1,0),\ (0,1),\ (-1,-1)$ Facets : $\begin{cases} -x-y \geq -1 \\ 2x-y \geq -1 \\ -x+2y \geq -1 \end{cases}$
Δ_2°		Vertices : $(-1,2),\ (-1,-1),\ (2,-1)$ Facets : $\begin{cases} -x-y \geq -1 \\ x \geq -1 \\ y \geq -1 \end{cases}$

THM: Reflexive $\Leftrightarrow$ single interior lattice point

(set to origin; all facets = hyperplanes of distance 1 away)

Toric Variety from Δ_n

- Face Fan $\Sigma(\Delta) \equiv \{\sigma = \text{pos}(F) \big| F \in \text{Faces}(\Delta)\}$ with
  $\text{pos}(F) \equiv \left\{\sum_i \lambda_i \underline{v}_i \big| \underline{v}_i \in F\ ,\ \lambda_i \geq 0\right\}$
- e.g. $\Delta_2 =$  $\Rightarrow \Sigma(\Delta_2) =$ 

- $\Sigma(\Delta_n)$ then defines a compact Toric variety $X(\Delta_n)$ of $\dim_{\mathbb{C}} = n$
- $X(\Delta)$ called Gorenstein Fano, i.e., $-K_X$ is Cartier and ample, i.e., $\mathcal{O}(-K_X)$ is line bundle and X is positive curvature
- THM: $X(\Delta)$ smooth $\Leftrightarrow$ generators of every cone σ is part of $\mathbb{Z}$-basis, i.e., $\det(\text{gens}(\sigma)) = \pm 1$ Back to KS CY3

Observatio Curiosa

- Penn group *purely abstract*, but $X_0^{19,19} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}$, Tian-Yau: $\begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$
- TRANSPOSES!!
- Why should the best manifold from 80's be so-simply related to the best manifold from completely different data-set and construction 20 years later ??
- Two manifolds are conifold transitions and vector bundles thereon transgress to one another ([Candelas-de la Ossa-YHH-Szendroi, 2008])
- **Connectedness of the Heterotic Landscape**
  - All CICY's are related by conifold transitions
  - Reid Conjecture: All CY3 are connected
  - Proposal: All (stable) vector bundles on all CY3 transgress

Back to Compactifications

A Computational Approach

- Northeastern/Witts/Notre Dame/Cornell Collaboration: Programme to study the computational algebraic geometry of $\mathcal{M}$: joint with M. Stillman, D. Grayson, H. Schenck (Macaulay 2), J. Hauenstein (Bertini), B. Nelson, V. Jejjala

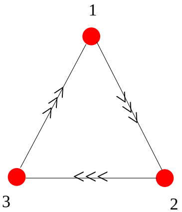
  1. n-fields: start with polynomial ring $\mathbb{C}[\phi_1, \ldots, \phi_n]$
  2. D = set of k GIO's: a ring map $\mathbb{C}[\phi_1, \ldots, \phi_n] \stackrel{D}{\longrightarrow} \mathbb{C}[D_1, \ldots, D_k]$
  3. Now incorporate superpotential: F-flatness $\langle f_{i=1,\ldots,n} = \frac{\partial W(\phi_i)}{\partial \phi_i} = 0 \rangle \simeq$ ideal of $\mathbb{C}[\phi_1, \ldots, \phi_k]$
  4. Moduli space = image of the ring map

$$\frac{\mathbb{C}[\phi_1,\ldots,\phi_n]}{\{F=\langle f_1,\ldots,f_n\rangle\}} \stackrel{D=GIO}{\longrightarrow} \mathbb{C}[D_1, \ldots, D_k], \quad \mathcal{M} \simeq \mathrm{Im}(D)$$

- Image is an ideal of $\mathbb{C}[D_1, \ldots, D_k]$, i.e., $\mathcal{M}$ explicitly realised as an affine variety in $\mathbb{C}^k$

Abelian Quotient: $\mathcal{M} = \mathbb{C}^3/\Gamma$

- All abelian orbifolds are toric.
- Archetypal example: $\mathbb{C}^3/\mathbb{Z}_3$ with action $(1,1,1) \rightsquigarrow U(1)^3$ quiver theory



$$W = \epsilon_{\alpha\beta\gamma} X_{12}^{(\alpha)} X_{23}^{(\beta)} X_{31}^{(\gamma)}, \qquad X_{12}^{(\alpha)}, X_{23}^{(\beta)}, X_{31}^{(\gamma)}, \alpha, \beta, \gamma = 1, 2, 3$$

Adjacency Matrix: $A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$

Incidence Matrix: $d = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}$

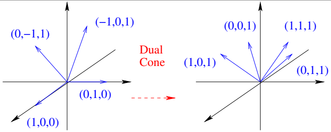
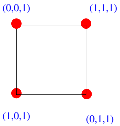
- loops: $3^3 = 27$ GIOs; arrows: 3×3 fields
- Moduli space: 27 quadrics in $\mathbb{C}^{10}$, explicit equations for

  $\mathbb{C}^3/\mathbb{Z}_3 \simeq Tot(\mathcal{O}_{\mathbb{P}^2}(-3))$

Back to Toric Quivers

Notation for Affine Toric Variety

Back to Toric Quivers

Def		Example (Conifold)
Comb.:	Convex Cone $\sigma \in \mathbb{Z}^d \rightsquigarrow$ Dual Cone $\sigma^\vee \rightsquigarrow X =$ $\text{Spec}_{Max}\mathbb{C}[S_\sigma = x_i^{\text{gen}(\sigma^\vee)\cap\mathbb{Z}^d}]$ Toric Diagram $= S_\sigma$	 $S_\sigma = \langle a = z, c = yz, b = xyz, d = xz\rangle$ $ab = cd$ in $\mathbb{C}^4[a,b,c,d]$
Symp:	Generalise $\mathbb{P}^n$: a $(\mathbb{C}^*)^{q-d}$ action on $\mathbb{C}^q_{[x_i]}$ $x_i \mapsto \lambda_a^{Q_{i=1\ldots q}^{a=1\ldots q-d}} x_i$ with Relations: $\sum\limits_{i=1}^{d} Q_i^a v_i = 0$ Toric Diagram $= v_i$	 $Q = [-1,-1,1,1]$ $\mathbb{C}^*$ on $\mathbb{C}^4 \rightsquigarrow$ $\ker Q = G_t =$ $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$
Comp:	Binomial Ideal $\langle \prod p_i = \prod q_j \rangle$	$ab = cd$ in $\mathbb{C}^4$

Tropical Geomtry: Amoebae & Algae

- Amoeba Projection $Log(z,w) \to (\log|z|, \log|w|)$

$$A = Amoeba(P(z,w) \subset (\mathbb{C}^*)^2) = Log(P) \subset \mathbb{R}^2 \rightsquigarrow$$

skeleton of A is the (p,q)-configuration

- T^2 of dimer model lives in the T^3 of mirror symmetry
  - $P(z,w) = 0$ describes fiber Σ over $s = 0$ in mirror CY3
  - ($\bigcap$ 3-cycles)$\cap\Sigma$ at a graph Γ on $T^2 \subset T^3 \rightsquigarrow$ periodic tiling
  - Alga Projection: $Arg(z,w) \to (\arg(z), \arg(w))$

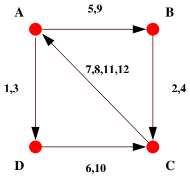
$$Alga(P(z,w) \subset (\mathbb{C}^*)^2) = Arg(P) \subset [0, 2\pi)^2 \rightsquigarrow$$

fundamental region of dimer

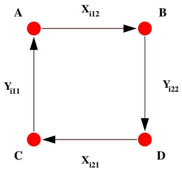
Back to Toric Quivers

Toric/Quiver/Seiberg Duality: Plethora of Examples

F0

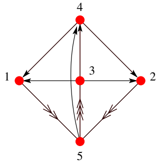


Model I

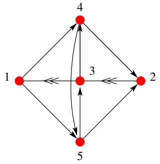


Model II

dP2

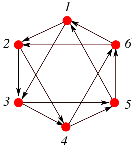


Model I

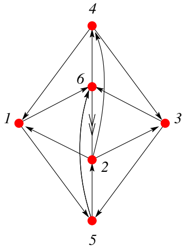


Model II

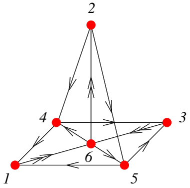
dP3



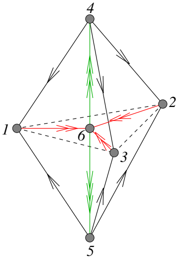
Model I



Model II



Model III

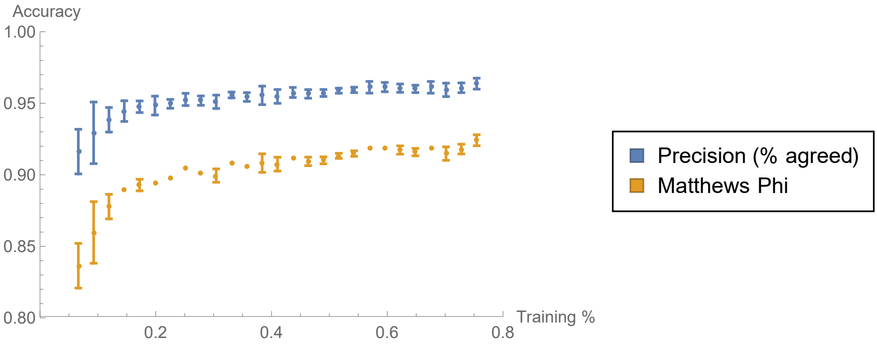


Model IV

Perspectives on Seiberg Duality

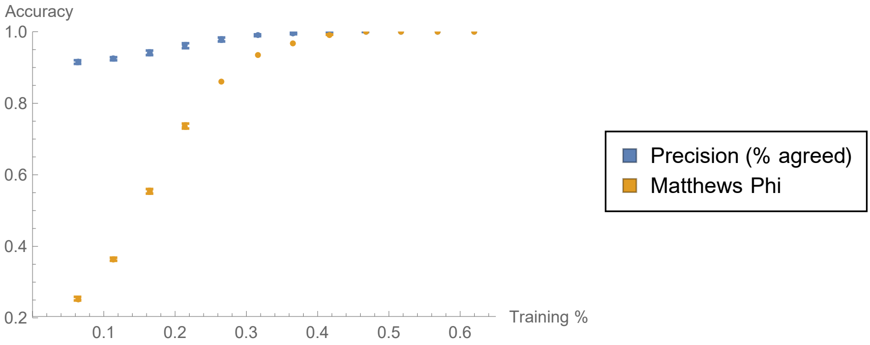
- Mirror Picture $Fuk(Y)$ (Type IIA)
  - D6-branes wrapping SL-$k+3$ cycles S_i in the mirror Y
  - Quiver $=$ intersection matrix $A_{ij} = S_i \circ S_j$
  - **Picard-Lefschetz** $S_i \to S_i - (S_i \circ S_{i_0}) S_{i_0}$
- Derived Category $D^\flat(X)$ (Type IIB)
  - think of brane as support for coherent sheaf w/ $\mathrm{ch}(F_i) := (\mathrm{rk}, c_1, c_2)$
  - Quiver: $A_{ij} = \chi(F_i, F_j) := \sum\limits_m (-1)^m \dim_{\mathbb{C}} \mathrm{Ext}^m(F_i, F_j)$
  - mutation of exceptional collection of F_i
- Cluster Algebra
  - cluster mutation rules on cluster (matrix) variables
  - Gadde-Gukov-Putrov, Franco-Lee-Seong-Vafa, other dim.
  - relation to total positivity and Grassmannian? (cf. Arkani-Hamed, Cachazo, Bourjaily, Trnka et al.; Franco (BFT)) Back to Tilings

Learning Curve: WP4



Return

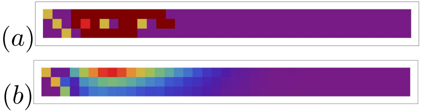
Learning Curve: CICY



Return

KS Dataset: *Gradus ad Parnasum*

- 4319 reflexive Δ_3 correspond to compact K3 surfaces or non-compact CY3
- Each is an integer matrix (padded) 3×39 with entries in $[0, 28]$, pixelate with 28 shades of colour



(a) typical Δ_3;

(b) average Δ_3

- Data size not so big for $n = 3$; training against for example, Sasaki-Einstein Volume or Picard Number achieves $\sim 60\%$ accuracy in a few minutes
- **GOAL:** to learn from geometrical quantities in a subset of $\sim 10^{5-6}$ (currently within computer power) to predict the full $\sim 10^{10}$ Δ_4 (currently beyond computer power) (to do ...)

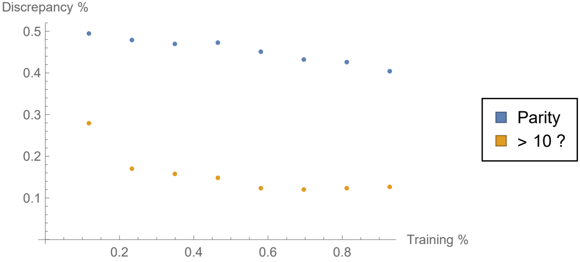
Toric Quiver Gauge Theories

- Infinite number of theories: any convex lattice polygon $\rightsquigarrow$ non-compact CY3 which D3-brane can probe; 2 databases so far:
  - Davey-Hanany-Pasukonis, 2009 (by terms in superpotential);
  - updated and expanded Chuang-Franco-YHH-Xiao, 2017 (by area of polygon)
- **computationally hard:** finding dual cone exponential-running; even with dimer/brane-tiling technology, Higgsing/perfect-matchings time-consuming
- Try on dataset1, (small) size = 375
  - INPUT: combined integer matrix Q_{DF}: incidence matrix from D-terms; exponent matrix from F-terms
  - OUTPUT: e.g., # gauge groups (train 100, predicts to $\sim 97\%$) Learning Curves
- TO DO: use this to **predict** unknown gauge theory given big toric diagrams

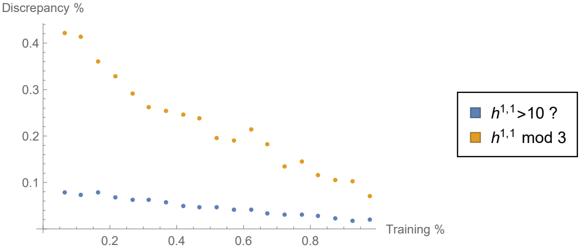
Back to CICYs

Learning Curves

Picard Numbers of K3 hyperfaces in toric Fano 3-folds from reflexive Δ_3



$h^{1,1}$ of CICYs:



Return