$SL(2, \mathbb{C})$-Chern-Simons theory and the AJ conjecture

Geometry, Quantum Topology and Asymptotics 2018

Alessandro Malusà

Aarhus University and Centre for Quantum Geometry of Moduli Spaces
Pacific Institute for the Mathematical Sciences and Uiversity of Saskatchewan

July 11th, 2018

Joint with Jørgen Ellegaard Andersen
arXiv: 1711.11522

The partition function of the Teichmüller TQFT

The partition function of the Teichmüller TQFT

Let $N \in \mathbb{Z}_{>0}$ be odd, $\mathrm{b} \in S^1 \subseteq \mathbb{C}$ with positive imaginary part, and set

$$\mathbb{A}_N = \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}\,, \qquad \mathbb{A}_N^{\mathbb{C}} = \mathbb{C} \times \mathbb{Z}/N\mathbb{Z}\,,$$

as measured spaces.

The partition function of the Teichmüller TQFT

Let $N \in \mathbb{Z}_{>0}$ be odd, $\mathrm{b} \in S^1 \subseteq \mathbb{C}$ with positive imaginary part, and set

$$\mathbb{A}_N = \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}\,, \qquad \mathbb{A}_N^{\mathbb{C}} = \mathbb{C} \times \mathbb{Z}/N\mathbb{Z}\,,$$

as measured spaces.

Conjecture (Andersen-Kashaev 2014)

Let M be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subseteq M$, there exists a two-parameter (b, N) family of smooth functions $J_{M,K}^{(\mathrm{b},N)}(\mathbf{x})$ on $\mathbb{A}_N$ which enjoys the following properties. For any fully balanced shaped ideal triangulation X of the complement of K in M, there exist a gauge-invariant real linear combination of dihedral angles λ, and a (gauge-dependent) real quadratic polynomial of dihedral angles ϕ, such that

$$\mathcal{Z}_{\mathrm{b}}^{(N)}(X) = e^{ic_{\mathrm{b}}^2\phi} \int_{\mathbb{A}_N} J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) e^{i\lambda c_{\mathrm{b}} x}\, \mathrm{d}\mathbf{x}\,.$$

The partition function of the Teichmüller TQFT

Let $N \in \mathbb{Z}_{>0}$ be odd, $\mathrm{b} \in S^1 \subseteq \mathbb{C}$ with positive imaginary part, and set

$$\mathbb{A}_N = \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}\,, \qquad \mathbb{A}_N^{\mathbb{C}} = \mathbb{C} \times \mathbb{Z}/N\mathbb{Z}\,,$$

as measured spaces.

Conjecture (Andersen-Kashaev 2014)

Let M be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subseteq M$, there exists a two-parameter (b, N) family of smooth functions $J_{M,K}^{(\mathrm{b},N)}(\mathbf{x})$ on $\mathbb{A}_N$ which enjoys the following properties. For any fully balanced shaped ideal triangulation X of the complement of K in M, there exist a gauge-invariant real linear combination of dihedral angles λ, and a (gauge-dependent) real quadratic polynomial of dihedral angles ϕ, such that

$$\mathcal{Z}_{\mathrm{b}}^{(N)}(X) = e^{ic_{\mathrm{b}}^2\phi} \int_{\mathbb{A}_N} J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) e^{i\lambda c_{\mathrm{b}} x}\, \mathrm{d}\mathbf{x}\,.$$

[...asymptotic properties...]

Two examples: 4_1 (the figure-eight) and 5_2

Two examples: 4_1 (the figure-eight) and 5_2

$$J^{(\mathrm{b},N)}_{S^3,4_1}(\mathbf{x}) = e^{4\pi i \frac{c_\mathrm{b} x}{\sqrt{N}}} \int_{\mathbb{A}_N} \frac{\varphi_\mathrm{b}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_\mathrm{b}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}} \, d\mathbf{y} \,,$$

$$J^{(\mathrm{b},N)}_{S^3,5_2}(\mathbf{x}) = e^{2\pi i \frac{c_\mathrm{b} x}{\sqrt{N}}} \int_{\mathbb{A}_N} \frac{\langle \mathbf{y}\rangle \, \langle \mathbf{x}\rangle^{-1}}{\varphi_\mathrm{b}(\mathbf{y}+\mathbf{x})\varphi_\mathrm{b}(\mathbf{y})\varphi_\mathrm{b}(\mathbf{y}-\mathbf{x})} \, d\mathbf{y} \,.$$

Two examples: 4_1 (the figure-eight) and 5_2

$$J^{(\mathrm{b},N)}_{S^3,4_1}(\mathbf{x}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \int_{\mathbb{A}_N} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}} \, d\mathbf{y}\,,$$

$$J^{(\mathrm{b},N)}_{S^3,5_2}(\mathbf{x}) = e^{2\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \int_{\mathbb{A}_N} \frac{\langle \mathbf{y}\rangle \, \langle \mathbf{x}\rangle^{-1}}{\varphi_{\mathrm{b}}(\mathbf{y}+\mathbf{x})\varphi_{\mathrm{b}}(\mathbf{y})\varphi_{\mathrm{b}}(\mathbf{y}-\mathbf{x})} \, d\mathbf{y}\,.$$

where φ_{b} is the level-N quantum dilogarithm.

Quantum dilogarithm

Quantum dilogarithm

$$\varphi_{\mathrm{b}}\left(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\right) = \left(1 - e^{-\frac{\mathrm{b}^2+1}{N}} e^{2\pi \frac{\mathrm{b}}{\sqrt{N}} x} e^{-2\pi i \frac{n}{N}}\right) \varphi_{\mathrm{b}}(\mathbf{x}) \, .$$

Quantum dilogarithm

$$\varphi_{\mathrm{b}}\left(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\right) = \left(1 - e^{-\frac{\mathrm{b}^2+1}{N}} e^{2\pi \frac{\mathrm{b}}{\sqrt{N}} x} e^{-2\pi i \frac{n}{N}}\right) \varphi_{\mathrm{b}}(\mathbf{x}) \,.$$

Define operators acting on meromorphic functions on $\mathbb{A}_N^{\mathbb{C}}$

$$\widehat{m}_{\mathbf{x}} f(\mathbf{x}) = e^{-2\pi \frac{\mathrm{b}x}{\sqrt{N}}} e^{2\pi i \frac{n}{N}} f(\mathbf{x}) \,, \qquad \widehat{\ell}_{\mathbf{x}} f(x, n) := f\left(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\right),$$

Quantum dilogarithm

$$\varphi_{\mathrm{b}}\left(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\right) = \left(1 - e^{-\frac{\mathrm{b}^2+1}{N}} e^{2\pi \frac{\mathrm{b}}{\sqrt{N}} x} e^{-2\pi i \frac{n}{N}}\right) \varphi_{\mathrm{b}}(\mathbf{x}) .$$

Define operators acting on meromorphic functions on $\mathbb{A}_N^{\mathbb{C}}$

$$\widehat{m}_{\mathbf{x}} f(\mathbf{x}) = e^{-2\pi \frac{\mathrm{b}x}{\sqrt{N}}} e^{2\pi i \frac{n}{N}} f(\mathbf{x}) , \qquad \widehat{\ell}_{\mathbf{x}} f(x, n) := f\left(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\right) ,$$

so

$$\widehat{\ell}_{\mathbf{x}} \varphi_{\mathrm{b}}(\mathbf{x}) = \left(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1}\right) \varphi_{\mathrm{b}}(\mathbf{x}) \quad \text{for } q^{\frac{1}{2}} := -e^{\pi i \frac{\mathrm{b}^2+1}{N}} .$$

Theorem (Weil-Gel'fand-Zak transform)

There exists a unitary isomorphism

$$\mathcal{H} \to \mathsf{L}^2(\mathbb{A}_N) ,$$

mapping the quantum operators associated to m and ℓ to $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$, respectively.

Complex $\widehat{A}$-polynomial

Complex $\widehat{A}$-polynomial

If $f\colon \mathbb{A}_N^{\mathbb{C}} \to \mathbb{C}$ is meromorphic, one can define a special element $\widehat{A}_f^{\mathbb{C}} \in \mathcal{I}(f)$.

Complex $\widehat{A}$-polynomial

If $f\colon \mathbb{A}_N^{\mathbb{C}} \to \mathbb{C}$ is meromorphic, one can define a special element $\widehat{A}_f^{\mathbb{C}} \in \mathcal{I}(f)$. Suppose that for a knot $K \subseteq M$ one has

$$J_{M,K}^{(\mathrm{b},N)}(\mathsf{x}) = \int_{\mathbb{A}_N^h} \Phi(\mathsf{x}, \mathsf{x}_1, \ldots, \mathsf{x}_h)\, \mathrm{d}\mathsf{x}_1 \ldots \mathrm{d}\mathsf{x}_h \,,$$

Φ a product of dilogarithms and exponentials of degree-2 polynomials.

Complex $\widehat{A}$-polynomial

If $f \colon \mathbb{A}_N^{\mathbb{C}} \to \mathbb{C}$ is meromorphic, one can define a special element $\widehat{A}_f^{\mathbb{C}} \in \mathcal{I}(f)$. Suppose that for a knot $K \subseteq M$ one has

$$J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) = \int_{\mathbb{A}_N^h} \Phi(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_h)\, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_h \,,$$

Φ a product of dilogarithms and exponentials of degree-2 polynomials.

- For each $j = 0, \ldots, h$ find a rational function p_j such that

$$\widehat{\ell}_j \Phi = p_j(\widehat{m}_{\mathbf{x}}, \widehat{m}_1, \ldots, \widehat{m}_h)\Phi \,;$$

Complex $\widehat{A}$-polynomial

If $f\colon \mathbb{A}_N^{\mathbb{C}} \to \mathbb{C}$ is meromorphic, one can define a special element $\widehat{A}_f^{\mathbb{C}} \in \mathcal{I}(f)$. Suppose that for a knot $K \subseteq M$ one has

$$J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) = \int_{\mathbb{A}_N^h} \Phi(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_h)\, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_h \,,$$

Φ a product of dilogarithms and exponentials of degree-2 polynomials.

- For each $j = 0, \ldots, h$ find a rational function p_j such that

$$\widehat{\ell}_j \Phi = p_j(\widehat{m}_{\mathbf{x}}, \widehat{m}_1, \ldots, \widehat{m}_h)\Phi \,;$$

- Give common denominators and find polynomials g_j in $\widehat{m}_j, \widehat{\ell}_j$;

Complex $\widehat{A}$-polynomial

If $f\colon \mathbb{A}_N^{\mathbb{C}} \to \mathbb{C}$ is meromorphic, one can define a special element $\widehat{A}_f^{\mathbb{C}} \in \mathcal{I}(f)$. Suppose that for a knot $K \subseteq M$ one has

$$J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) = \int_{\mathbb{A}_N^h} \Phi(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_h)\, \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_h \,,$$

Φ a product of dilogarithms and exponentials of degree-2 polynomials.

- For each $j = 0, \ldots, h$ find a rational function p_j such that

$$\widehat{\ell}_j \Phi = p_j(\widehat{m}_{\mathbf{x}}, \widehat{m}_1, \ldots, \widehat{m}_h)\Phi \,;$$

- Give common denominators and find polynomials g_j in $\widehat{m}_j, \widehat{\ell}_j$;
- Eliminate the variables $\widehat{m}_1, \ldots \widehat{m}_h$ to find a polynomial in $\widehat{m}_{\mathbf{x}}$, $\widehat{\ell}_{\mathbf{x}}$, $\widehat{\ell}_1, \ldots, \widehat{\ell}_h$ annihilating Φ;

Complex $\widehat{A}$-polynomial

If $f \colon \mathbb{A}_N^{\mathbb{C}} \to \mathbb{C}$ is meromorphic, one can define a special element $\widehat{A}_f^{\mathbb{C}} \in \mathcal{I}(f)$. Suppose that for a knot $K \subseteq M$ one has

$$J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) = \int_{\mathbb{A}_N^h} \Phi(\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_h)\, \mathrm{d}\mathbf{x}_1 \dots \mathrm{d}\mathbf{x}_h \,,$$

Φ a product of dilogarithms and exponentials of degree-2 polynomials.

- For each $j = 0, \dots, h$ find a rational function p_j such that

$$\widehat{\ell}_j \Phi = p_j(\widehat{m}_{\mathbf{x}}, \widehat{m}_1, \dots, \widehat{m}_h)\Phi \,;$$

- Give common denominators and find polynomials g_j in $\widehat{m}_j, \widehat{\ell}_j$;
- Eliminate the variables $\widehat{m}_1, \dots \widehat{m}_h$ to find a polynomial in $\widehat{m}_{\mathbf{x}}$, $\widehat{\ell}_{\mathbf{x}}$, $\widehat{\ell}_1, \dots, \widehat{\ell}_h$ annihilating Φ;
- *Carefully* evaluate $\widehat{\ell}_j = 1$ and take the polynomial out of the integral.

The figure-eight knot

The figure-eight knot

$$\Phi(\mathbf{x},\mathbf{y}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}}\,,$$

The figure-eight knot

$$\Phi(\mathbf{x},\mathbf{y}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y}) \langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y}) \langle \mathbf{y} \rangle^{-2}} ,$$

$$g_{\mathbf{x}} = \widehat{\ell}_{\mathbf{x}} \widehat{m}_{\mathbf{y}}^2 - q^{\frac{1}{2}} \widehat{m}_{\mathbf{x}} \widehat{m}_{\mathbf{y}} - q \widehat{m}_{\mathbf{x}}^2 ,$$

$$g_{\mathbf{y}} = \widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}} \widehat{m}_{\mathbf{y}}^2 + q^{\frac{1}{2}} \left(\widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}}^2 + \widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}} - q \right) \widehat{m}_{\mathbf{y}} + q \widehat{\ell}_{\mathbf{y}} \widehat{m}_{\mathbf{x}}^2 .$$

The figure-eight knot

$$\Phi(\mathbf{x},\mathbf{y}) = e^{4\pi i \frac{c_b x}{\sqrt{N}}} \frac{\varphi_b(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_b(\mathbf{y})\langle \mathbf{y}\rangle^{-2}}\,,$$

$$\begin{aligned}\widehat{A} = {} & q^2\widehat{\ell}_{\mathbf{y}}^2\Big(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\Big)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2 \\ & -\Big(q^2\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\Big)\Big(q^4\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^4 - q^3\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^3 - q(q^2+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} + 1\Big)\widehat{\ell}_{\mathbf{x}} \\ & +q^2\widehat{\ell}_{\mathbf{y}}\Big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\Big)\widehat{m}_{\mathbf{x}}^2\,.\end{aligned}$$

The figure-eight knot

$$\Phi(\mathbf{x},\mathbf{y}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y}) \langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y}) \langle \mathbf{y} \rangle^{-2}} \,,$$

$$\begin{aligned} \widehat{A}^{\mathbb{C}}_{4_1} &= q^2 \left(q \widehat{m}_{\mathbf{x}}^2 - 1 \right) \widehat{m}_{\mathbf{x}}^2 \widehat{\ell}_{\mathbf{x}}^2 \\ &\quad - \left(q^2 \widehat{m}_{\mathbf{x}}^2 - 1 \right) \left(q^4 \widehat{m}_{\mathbf{x}}^4 - q^3 \widehat{m}_{\mathbf{x}}^3 - q(q^2+1) \widehat{m}_{\mathbf{x}}^2 - q \widehat{m}_{\mathbf{x}} + 1 \right) \widehat{\ell}_{\mathbf{x}} \\ &\quad + q^2 \left(q^3 \widehat{m}_{\mathbf{x}}^2 - 1 \right) \widehat{m}_{\mathbf{x}}^2 \,, \end{aligned}$$

The figure-eight knot

$$\Phi(\mathbf{x},\mathbf{y}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}},$$

$$\begin{aligned}\widehat{A}^{\mathbb{C}}_{4_1} &= q^2\left(q\widehat{m}_{\mathbf{x}}^2 - 1\right)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2 \\ &\quad - \left(q^2\widehat{m}_{\mathbf{x}}^2 - 1\right)\left(q^4\widehat{m}_{\mathbf{x}}^4 - q^3\widehat{m}_{\mathbf{x}}^3 - q(q^2+1)\widehat{m}_{\mathbf{x}}^2 - q\widehat{m}_{\mathbf{x}} + 1\right)\widehat{\ell}_{\mathbf{x}} \\ &\quad + q^2\left(q^3\widehat{m}_{\mathbf{x}}^2 - 1\right)\widehat{m}_{\mathbf{x}}^2,\end{aligned}$$

$$\begin{aligned}\widehat{A}^{\mathrm{nh}}_{q,4_1} &= q^2Q^2\left(q^2Q - 1\right)\left(qQ^2 - 1\right)E^2 \\ &\quad -\left(q^2Q^2 - 1\right)\left(qQ - 1\right)\left(q^4Q^4 - q^3Q^3 - q(q^2+1)Q^2 - qQ + 1\right)E \\ &\quad +q^2Q^2\left(Q - 1\right)\left(q^3Q^2 - 1\right),\end{aligned}$$

The figure-eight knot

$$\Phi(\mathbf{x},\mathbf{y}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}} ,$$

$$\begin{aligned} \widehat{A}^{\mathbb{C}}_{4_1} &= q^2\Big(q\widehat{m}_{\mathbf{x}}^2 - 1\Big)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2 \\ &\quad - \Big(q^2\widehat{m}_{\mathbf{x}}^2 - 1\Big)\Big(q^4\widehat{m}_{\mathbf{x}}^4 - q^3\widehat{m}_{\mathbf{x}}^3 - q(q^2+1)\widehat{m}_{\mathbf{x}}^2 - q\widehat{m}_{\mathbf{x}} + 1\Big)\widehat{\ell}_{\mathbf{x}} \\ &\quad + q^2\Big(q^3\widehat{m}_{\mathbf{x}}^2 - 1\Big)\widehat{m}_{\mathbf{x}}^2 , \end{aligned}$$

$$\begin{aligned} \widehat{A}^{\mathrm{nh}}_{q,4_1} &= q^2Q^2\Big(q^2Q - 1\Big)\Big(qQ^2 - 1\Big)E^2 \\ &\quad -\Big(q^2Q^2 - 1\Big)\Big(qQ - 1\Big)\Big(q^4Q^4 - q^3Q^3 - q(q^2+1)Q^2 - qQ + 1\Big)E \\ &\quad +q^2Q^2\Big(Q - 1\Big)\Big(q^3Q^2 - 1\Big) , \end{aligned}$$

$$A_{4_1}(m,\ell) = m^4\ell^2 - \Big(m^8 - m^6 - 2m^4 - m^2 + 1\Big)\ell + m^4 .$$

The AJ conjecture for the Teichmüller TQFT

The AJ conjecture for the Teichmüller TQFT

Conjecture

Let $K \subseteq M$ be a knot inside a closed, oriented 3-manifold, with hyperbolic complement. Then the non-commutative polynomial $\widehat{A}_K^{\mathbb{C}}$ agrees with $\widehat{A}_K$ up to a right factor, linear in $\widehat{m}_{\mathbf{x}}$, and it reproduces the classical A-polynomial in the sense of the original AJ-conjecture.

Thank you...

...for your attention!