Infinite gap Jacobi matrices

Jacob Stordal Christiansen

Centre for Mathematical Sciences, Lund University

Spectral Theory of Orthogonal Polynomials
Master class by Barry Simon

Centre for Quantum Geometry of Moduli Spaces
Århus University

Outline

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- Parreau–Widom sets
  – a large class of compact subsets of the real line $\mathbb{R}$

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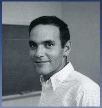
  – a large class of compact subsets of the real line $\mathbb{R}$
- Szegő class theory
  – including asymptotics of the orthogonal polynomials

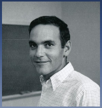
Literature

- M. Sodin and P. Yuditskii. Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions. *J. Geom. Anal.* **7** (1997) 387-435
- F. Peherstorfer and P. Yuditskii. Asymptotic behavior of polynomials orthonormal on a homogeneous set. *J. Anal. Math.* **89** (2003) 113–154
- J. S. Christiansen. Szegő's theorem on Parreau–Widom sets. *Adv. Math.* **229** (2012) 1180–1204
- C. Remling. The absolutely continuous spectrum of Jacobi matrices. *Ann. Math.* **174** (2011) 125–171
- P. Yuditskii. On the Direct Cauchy Theorem in Widom domains: Positive and negative examples. *Comput. Methods Funct. Theory* **11**, 395–414 (2011)
- J. S. Christiansen. Dynamics in the Szegő class and polynomial asymptotics.



Parreau–Widom sets

Let $\mathsf{E} \subset \mathbb{R}$ be a compact set of positive logarithmic capacity and denote by g the Green's function for $\Omega = \overline{\mathbb{C}} \setminus \mathsf{E}$ with pole at ∞.



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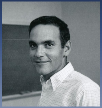

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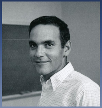
Recall that

$$g(x) = \int \log|t-x|\, d\mu_{\mathsf{E}}(t) - \log(\mathrm{Cap}(\mathsf{E}))$$

and write

$$\mathsf{E} = [\alpha, \beta] \setminus \bigcup_j (\alpha_j, \beta_j).$$



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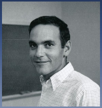
While g vanishes on E, it is concave on (α_j, β_j) for each j. So there is precisely one critical point c_j per gap.



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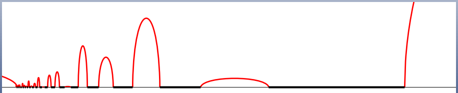

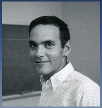

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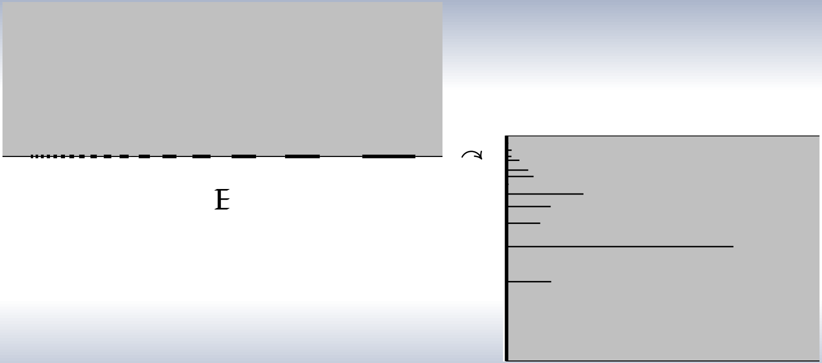
Defn. We call E a *Parreau–Widom set* if $\sum_j g(c_j) < \infty$.

Comb-like domains



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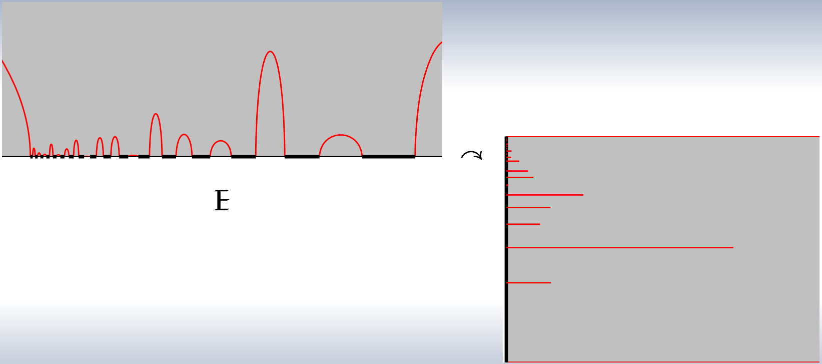

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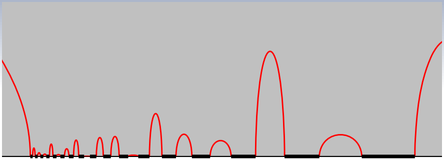

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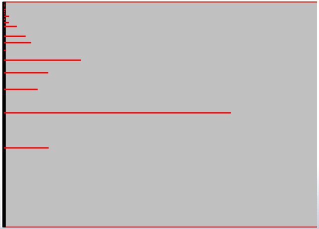

$d\mu_{\mathrm{E}}$ is absolutely continuous iff

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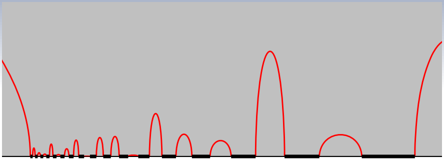

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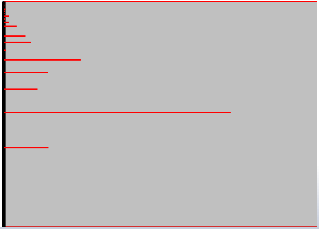

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This is always the case when $\sum_j g(c_j) < \infty$ (i.e., E is PW).

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It includes all compact sets E that are *homogeneous* in the sense of Carleson. By definition, this means there is an $\varepsilon > 0$ so that

$$\frac{|(t-\delta, t+\delta) \cap \mathrm{E}|}{\delta} \geq \varepsilon \ \text{ for all } t \in \mathrm{E} \text{ and all } \delta < \mathrm{diam}(\mathrm{E}).$$

This geometric condition was introduced to avoid cases where certain parts of E are very thin compared to Lebesgue measure.

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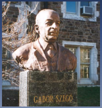
Example

Remove the middle $1/4$ from $[0, 1]$ and continue removing subintervals of length $1/4^n$ from the middle of each of the 2^{n-1} remaining intervals.

Let E be the set of what is left in $[0, 1]$ — a fat Cantor set of $|E| = 1/2$.

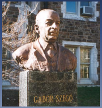
One can show that $|(t-\delta, t+\delta) \cap E| \geq \delta/4$ for all $t \in E$ and all $\delta < 1$.

Gábor Szegő



Szegő's theorem on PW sets

Let $\mathsf{E} \subset \mathbb{R}$ be a Parreau–Widom set and let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be a Jacobi matrix with spectral measure $d\rho = f(t)dt + d\rho_{\mathrm{s}}$.



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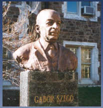
Assume that $\sigma_{\mathrm{ess}}(J) = \mathsf{E}$ and denote by $\{x_k\}$ the eigenvalues of J outside E, if any. [Adv. Math. 2012]

On condition that $\sum_k g(x_k) < \infty$, we have

$$\int_{\mathsf{E}} \log f(t)\, d\mu_{\mathsf{E}}(t) > -\infty \quad \Leftrightarrow \quad \limsup_{n\to\infty} \frac{a_1 \cdots a_n}{\mathrm{Cap}(\mathsf{E})^n} > 0.$$

In the affirmative,

$$0 < \liminf_{n\to\infty} \frac{a_1 \cdots a_n}{\mathrm{Cap}(\mathsf{E})^n} \leq \limsup_{n\to\infty} \frac{a_1 \cdots a_n}{\mathrm{Cap}(\mathsf{E})^n} < \infty.$$



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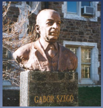
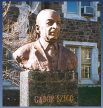
– What can be said about b_n and can we say more about a_n?



The Szegő class

Inspired by Szegő's theorem, we introduce the class $\mathrm{Sz}(\mathsf{E})$.



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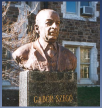

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– Do we have power asymptotics of the orthogonal polynomials ?

Reflectionless operators

Given a PW set E, we denote by $\mathcal{T}_{\mathrm{E}}$ the set of all two-sided Jacobi matrices $J' = \{a'_n, b'_n\}_{n=-\infty}^{\infty}$ that have spectrum equal to E and are *reflectionless* on E, that is,

$$\mathrm{Re}\left\langle \delta_n, \left(J' - (t + i0)\right)^{-1} \delta_n \right\rangle = 0 \text{ for a.e. } t \in \mathrm{E} \text{ and all } n.$$

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Equivalently,

$$(a'_n)^2 m_n^+(t+i0) = \frac{1}{\overline{m_n^-(t+i0)}} \ \text{ for a.e. } t \in \mathsf{E} \text{ and all } n,$$

where m_n^+ is the m-function for $J_n^+ = \{a'_{n+k}, b'_{n+k}\}_{k=1}^{\infty}$ and m_n^- the m-function for $J_n^- = \{a'_{n-k}, b'_{n+1-k}\}_{k=1}^{\infty}$.

Remling's theorem

By compactness, any bounded $J = \{a_n, b_n\}_{n=1}^{\infty}$ has accumulation points when the coefficients are shifted to the left.

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The theorem says that the left-shifts of J approach $\mathcal{T}_{\mathrm{E}}$ *as a set*.

Hence, $\mathcal{T}_{\mathrm{E}}$ is the natural limiting object associated with E.

The collection of divisors

Recall that

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The set $\mathcal{D}_{\mathsf{E}}$ of *divisors* consists of all formal sums

$$D = \sum_j (y_j, \pm), \quad y_j \in [\alpha_j, \beta_j],$$

where $(y_j, +)$ and $(y_j, -)$ are identified when y_j is equal to α_j or β_j.

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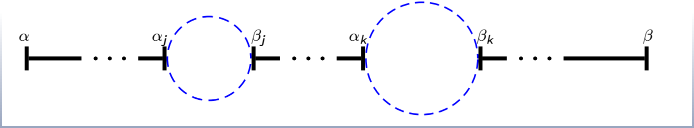

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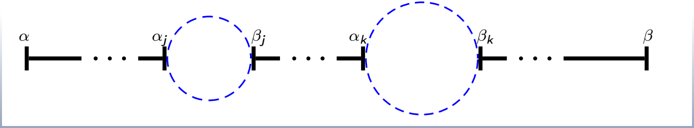

We shall equip $\mathcal{D}_{\mathsf{E}}$ with the product topology.

A map $\mathcal{T}_{\mathrm{E}} \to \mathcal{D}_{\mathrm{E}}$

When $J' \in \mathcal{T}_{\mathrm{E}}$, we know that $G(x) = \left\langle \delta_0, (J' - x)^{-1} \delta_0 \right\rangle$ is analytic on $\mathbb{C} \setminus \mathrm{E}$ and has purely imaginary boundary values a.e. on E.

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Such Nevanlinna–Pick functions admit a representation of the form

$$G(x) = \frac{-1}{\sqrt{(x-\alpha)(x-\beta)}} \prod_j \frac{x-y_j}{\sqrt{(x-\alpha_j)(x-\beta_j)}},$$

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Using the relation

$$(a_0')^2 m^+(x) - 1/m^-(x) = -1/G(x),$$

it follows that every $y_j \in (\alpha_j, \beta_j)$ is a pole of either m^+ or $1/m^-$.

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Such Nevanlinna–Pick functions admit a representation of the form

$$G(x) = \frac{-1}{\sqrt{(x-\alpha)(x-\beta)}} \prod_j \frac{x - y_j}{\sqrt{(x-\alpha_j)(x-\beta_j)}},$$

where $y_j \in [\alpha_j, \beta_j]$ for each j.

Using the relation

$$(a_0')^2 m^+(x) - 1/m^-(x) = -1/G(x),$$

it follows that every $y_j \in (\alpha_j, \beta_j)$ is a pole of either m^+ or $1/m^-$.

As m^+ and $1/m^-$ have no common poles, this in turn allows us to define a map $\mathcal{T}_{\mathrm{E}} \to \mathcal{D}_{\mathrm{E}}$.

Fuchsian groups & charaters

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Since an element in Γ_{E}^* is determined by its values on the generators of Γ_{E}, we can think of Γ_{E}^* as a compact torus (which is infinite dimensional when $\boldsymbol{\Omega}$ is infinitely connected).

The Abel map

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From our point of view, the map $\mathcal{T}_{\mathsf{E}} \to \Gamma_{\mathsf{E}}^*$ is given by the character of the *Jost function* of J (to be defined shortly).

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blaschke products

To every $w \in \mathbb{D}$, we associated the blaschke product

$$B(z,w) = \prod_{\gamma \in \Gamma_{\mathbf{E}}} \frac{|\gamma(w)|}{\gamma(w)} \frac{\gamma(w) - z}{1 - \overline{\gamma(w)} z}.$$

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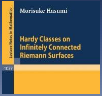
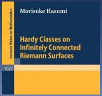
We mention in passing that $g\big(\mathbf{x}(z)\big) = -\log|B(z)|$.



The direct Cauchy theorem

As before, let $\{c_j\}$ denote the critical points of g (the Green's function for Ω with pole at ∞).



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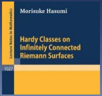

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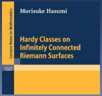

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Definition

The *direct Cauchy theorem* (DCT) is said to hold if

$$\int_0^{2\pi} \frac{\varphi(e^{i\theta})}{c(e^{i\theta})} \frac{d\theta}{2\pi} = \frac{\varphi(0)}{c(0)}$$

whenever $\varphi \in H^1(\mathbb{D})$ is character automorphic and $\chi_\varphi = \chi_c$.

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Let $\mathsf{E} \subset \mathbb{R}$ be a Parreau–Widom set and assume that the direct Cauchy theorem holds.

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Suppose that $J = \{a_n, b_n\}_{n=1}^{\infty}$ belongs to the Szegő class for E.

Then there is a unique $J' = \{a_n', b_n'\}_{n=-\infty}^{\infty}$ in $\mathcal{T}_{\mathsf{E}}$ such that

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Moreover, if $d\mu'$ is the spectral measure of J' restricted to $\ell^2(\mathbb{N})$, then

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Hence, $\prod(a_n/a'_n)$ and $\sum(b_n - b'_n)$ converge conditionally.

The Jost function

Let us introduce the key player for polynomial asymptotics.

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Given $J \in \mathrm{Sz}(\mathsf{E})$, we define the *Jost function* by

$$u(z;J) = \prod_k B(z,p_k) \exp\left\{-\frac{1}{2}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log f\big(\mathbf{x}(e^{i\theta})\big)\frac{d\theta}{2\pi}\right\},$$

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The Jost function is analytic on $\mathbb{D}$ and one can show that it is also *character automorphic*, that is,

$$\exists \chi \in \Gamma_{\mathsf{E}}^* : u(\gamma(z);J) = \chi(\gamma)\, u(z;J) \text{ for all } \gamma \in \Gamma_{\mathsf{E}}.$$

As for notation, we denote by $\chi(J)$ the character of J.

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3. the Jost asymptotics for right limits

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So $\chi(K) = \chi(K')$ and hence $K = K'$ by the Jost isomorphism.

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So we can write the diagonal Green's function as

$$G_{nn}(\mathbf{x}(z), d\mu) = P_{n-1}(\mathbf{x}(z), d\mu) u_n(z;J)/Wr(z),$$

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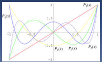
By use of the resolvent formula and since $J_n - J'_n \to 0$, the ratio of G_{nn}'s converges to 1. As J and J' have the same right limits, the ratio of u_n's also converges to 1.



Polynomial asymptotics

The proof shows that

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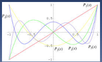
locally uniformly on $\mathbb{F}^{\mathrm{int}}$, a fundamental domain for Γ_{E}.



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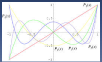

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In conclusion, we have

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Open problems

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Open question: Is it possible to characterize all ℓ^2-perturbations of $\mathcal{T}_{\mathsf{E}}$ through their spectral measures?

Thank you very much for your attention!