ORTHOGONAL POLYNOMIALS IN RANDOM MATRIX MODELS AND NON-INTERSECTING PATHS

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HOW TO GENERATE RANDOM POINTS?

Imagine we want to distribute on the real line several charges, assuming that their positions are random. This is the picture we get if we generate independent instances of random numbers:

But this picture does not reflect mutual repulsion. How can we built in this repulsion into the random picture?

That is what occupied Eugene Wigner in 1951, when he was looking for a model for the Hamiltonian of a heavy nucleus.

He was looking for something like this:

These are eigenvalues of a Hermitian random matrix.

Random Matrices?

ENSEMBLES OF RANDOM MATRICES

Random matrix theory $\approx$ branch of random spectral theory.

One of the **main problems**: asymptotics ($N \to \infty$) of various spectral characteristics of $N \times N$ matrices, whose probability distribution is given in terms of matrix elements.

$$\begin{pmatrix} M_{11} & M_{12} & \dots & M_{1N} \\ M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & \dots & M_{NN} \end{pmatrix}$$

Eigenvalues: $\lambda_1, \dots, \lambda_N$

Eigenvectors: $u_1, \dots, u_N$

Goal: "transfer" the probabilistic information from matrix elements to eigenvalues and eigenvectors.

Similarity with the spectral theory of Schrödinger operators with random potential.

ENSEMBLES OF RANDOM MATRICES

We are typically **given**:

- A class of $N \times N$ matrices $\mathcal{M}_N$ (e.g., Hermitian, Unitary, etc.).
- A probability measure $\boldsymbol{P}_N$ on $\mathcal{M}_N$.

The pair $(\mathcal{M}_N, \boldsymbol{P}_N)$ is a random matrix ensemble.

- A function $\Phi : \mathcal{M}_N \to \mathbb{R}$ or $\mathbb{C}$ (typically, orthogonal or unitary invariant).

We often need to calculate or estimate (as $N \to \infty$) values of integrals of the form

$$\int_{\mathcal{M}_N} \Phi(M_N) \boldsymbol{P}(dM_N)$$

GAUSSIAN UNITARY ENSEMBLE

First example:

$\mathcal{M}_N = \{N \times N \text{ Hermitian matrices}\}$

$\boldsymbol{P}_N$ Gaussian:

$$\boldsymbol{P}_N(dM) = \frac{1}{\widetilde{Z}_N} \exp\left(-\mathrm{Tr}\,(M^2)\right)\, dM$$

where

$$dM = \prod_{j=1}^{N} dM_{jj} \prod_{j \neq k}^{N} d\,\mathrm{Re}(M_{jk})\, d\,\mathrm{Im}(M_{jk})$$

(analogue of the Lebesgue measure on $\mathcal{M}_N$).

This is the Gaussian Unitary Ensemble (GUE).

We can consider more general probability distributions $\boldsymbol{P}_N$, by taking an arbitrary function (typically, a polynomial V), with

$$\boldsymbol{P}_N(dM) = \frac{1}{\widetilde{Z}_N} \exp\left(-\mathrm{Tr}\, V(M)\right)\, dM$$

UNITARY ENSEMBLES OF RANDOM MATRICES

$\mathcal{M}_N = \{N \times N \text{ Hermitian matrices}\}$

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$$\text{where } \; dM = \prod_{j=1}^{N} dM_{jj} \prod_{j \neq k}^{N} d\,\mathrm{Re}(M_{jk})\, d\,\mathrm{Im}(M_{jk})$$

According to the spectral theorem,

$$M = U \Lambda U^{-1}, \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$$

$$(M_{11}, \ldots, M_{NN}, \mathrm{Re}\, M_{12}, \mathrm{Im}\, M_{12}, \ldots) \mapsto (\lambda_1, \ldots, \lambda_N, u_{ij})$$

is a change of variables.

Weyl integration formula gives us its jacobian. According to it,

$$dM = c_n \prod_{i<j} (\lambda_j - \lambda_i)^2 d\lambda_1 \ldots d\lambda_N dU$$

dU is the Haar measure on the unitary group $U(n)$.

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This probability measure on matrices induces a joint probability density on their eigenvalues $\lambda_1 < \cdots < \lambda_N$:

$$\frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right),$$

with the corresponding partition function

$$Z_N = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right) d\lambda_1 \ldots d\lambda_N$$

Built-in repulsion

UNITARY ENSEMBLES OF RANDOM MATRICES

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The free energy of this matrix model is $F_N = -\dfrac{1}{N^2} \ln Z_N$

$$\boxed{F_N \sim F_\infty = ? \quad \text{as} \quad N \to \infty}$$

UNITARY ENSEMBLES OF RANDOM MATRICES

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$$Z_N = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right) d\lambda_1 \ldots d\lambda_N$$

Heuristics: $Z_N = \langle e^{-2E(\lambda_1,\ldots,\lambda_N)} \rangle$

$$E(\lambda_1, \ldots, \lambda_N) = \sum_{i<j} \log \frac{1}{|\zeta_i - \zeta_j|} + \frac{1}{2} \sum_{j=1}^{N} V(\lambda_j)$$

For large N, the expected distribution of λ_j's minimizes E.

This minimizer is the equilibrium measure in an external field given by V.

What about Orthogonal Polynomials?

CONNECTIONS WITH O. P.

There is another way to look at the expected distribution.

Recall the well-known formula by Heine ($\approx$ 1878): if μ is a measure, then

$$P_N(x) = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{j=1}^{N} (x-\lambda_j) \prod_{j<k} (\lambda_j - \lambda_k)^2 \, d\mu(\lambda_1) \dots d\mu(\lambda_N)$$

is orthogonal with respect to μ:

$$\int_{\mathbb{R}} P_N(x) x^k \, d\mu(x) = 0, \quad k = 0, 1, \dots, N-1.$$

Compare it with the probability measure of the eigenvalues,

$$\frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(- \sum_{i=1}^{N} V(\lambda_i) \right),$$

to conclude that the average characteristic polynomial of the unitary matrix ensemble is orthogonal with respect to $e^{-V(x)} dx$.

But not only that.

CONNECTIONS WITH O. P.

Observation:

Vandermonde

$$\prod_{i<j}(\lambda_i - \lambda_j) = \det \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^N \\ 1 & \lambda_2 & \dots & \lambda_2^N \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^N \end{pmatrix}$$

we can combine columns (almost) freely, replacing monomials in λ_j by monic polynomials!

Thus,

$$\det \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^N \\ 1 & \lambda_2 & \dots & \lambda_2^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^N \end{pmatrix} = \det \begin{pmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \dots & P_N(\lambda_1) \\ P_0(\lambda_1) & P_1(\lambda_2) & \dots & P_N(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_0(\lambda_1) & P_2(\lambda_N) & \dots & P_N(\lambda_N) \end{pmatrix}$$

Moreover,

$$(\det(A))^2 = \det(A)\det(A^*)$$

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we can combine columns (almost) freely, replacing monomials in λ_j by monic polynomials!

We conclude that if p_n are the orthogonal polynomials w.r.t. the weight $w(x) = \exp(-V(x))$, then (up to normalization),

$$\prod_{i<j}(\lambda_j - \lambda_i)^2 \exp\left(-\sum_{i=1}^{N} V(\lambda_j)\right) = \det(K_N(\lambda_i, \lambda_j))_{1\leq i,j\leq N},$$

where $K_N(x,y) = \sqrt{w(x)}\sqrt{w(y)}\sum_{j=0}^{N-1} p_j(x)p_j(y)$ is the correlation, reproducing or CD kernel.

This is an example of a determinantal point process.

CONNECTIONS WITH O. P.

Crucial property (Gram):

$$\int \dots \int \det \left[K_N(\lambda_j, \lambda_k)_{1 \le j,k \le N} \right] d\lambda_{d+1} \dots d\lambda_N$$

$$= (N-d)! \det \left[K_N(\lambda_j, \lambda_k)_{1 \le j,k \le d} \right]$$

Consequence: the partition function $\widehat{Z}_N$ and all statistics of the eigenvalue distribution can be expressed in terms of the CD kernel K_N.

In particular, for the large scale behavior, i.e. when $N \to \infty$, we must study the asymptotics of the corresponding orthogonal polynomials and their CD kernel (via de CD formula).

For instance, the density of the limit eigenvalue distribution is given by

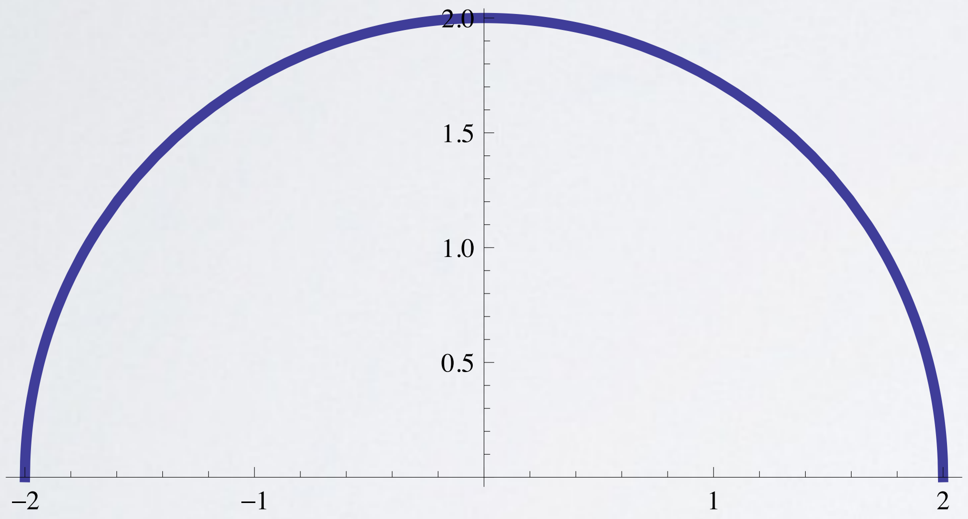
$$\rho(x) = \lim_{N \to \infty} \frac{1}{N} K_N(x, x).$$

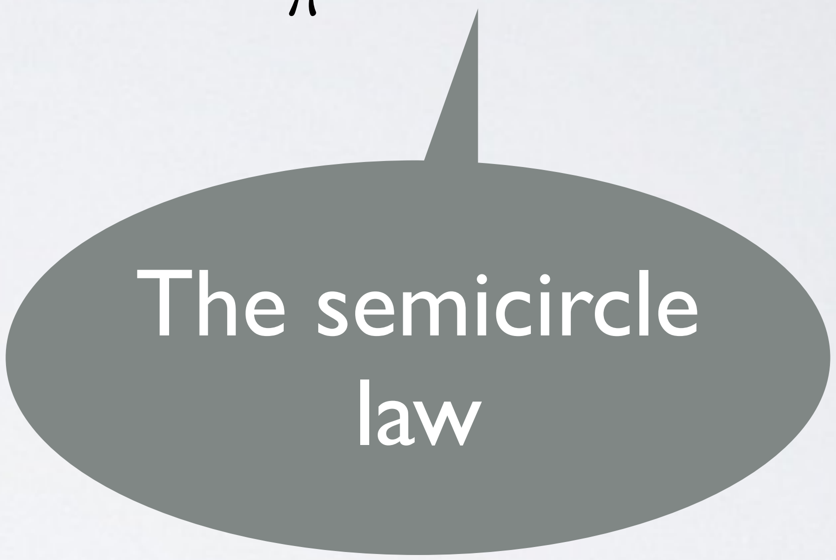
CONNECTIONS WITH O. P.

Case of GUE: $w(x) = e^{-x^2}$; or after $x \mapsto \sqrt{N}x$, $w_N(x) = e^{-Nx^2}$.

Hence, taking $p_N(x)$ as the Hermite polynomials we can compute:

$$\rho_{SC}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{N}} K_N(\sqrt{N}x, \sqrt{N}x) = \frac{1}{\pi}\sqrt{2 - x^2}, \quad |x| \leq \sqrt{2}$$





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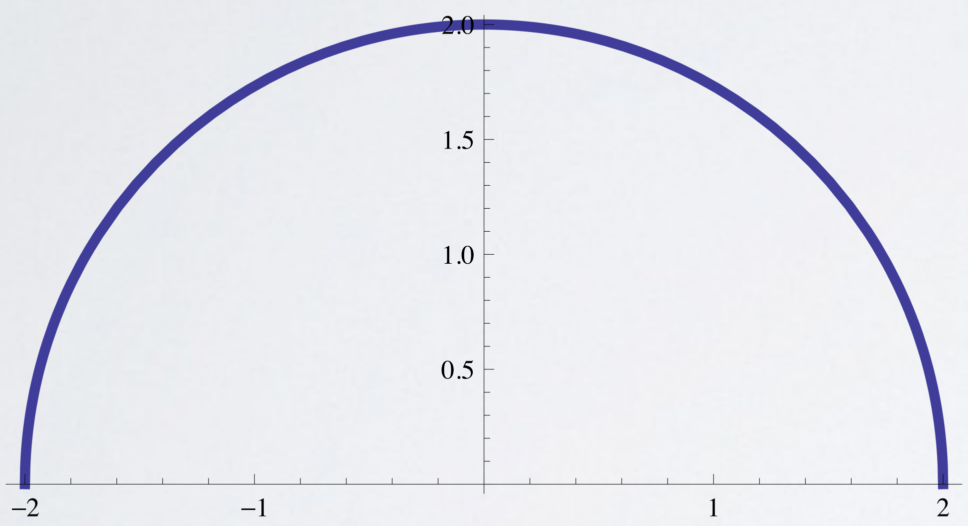

a.k.a. the equilibrium measure in the external field $x^2/2$

It gives us the "global picture"; now we can study the local fluctuations around different values (microscopic scale).

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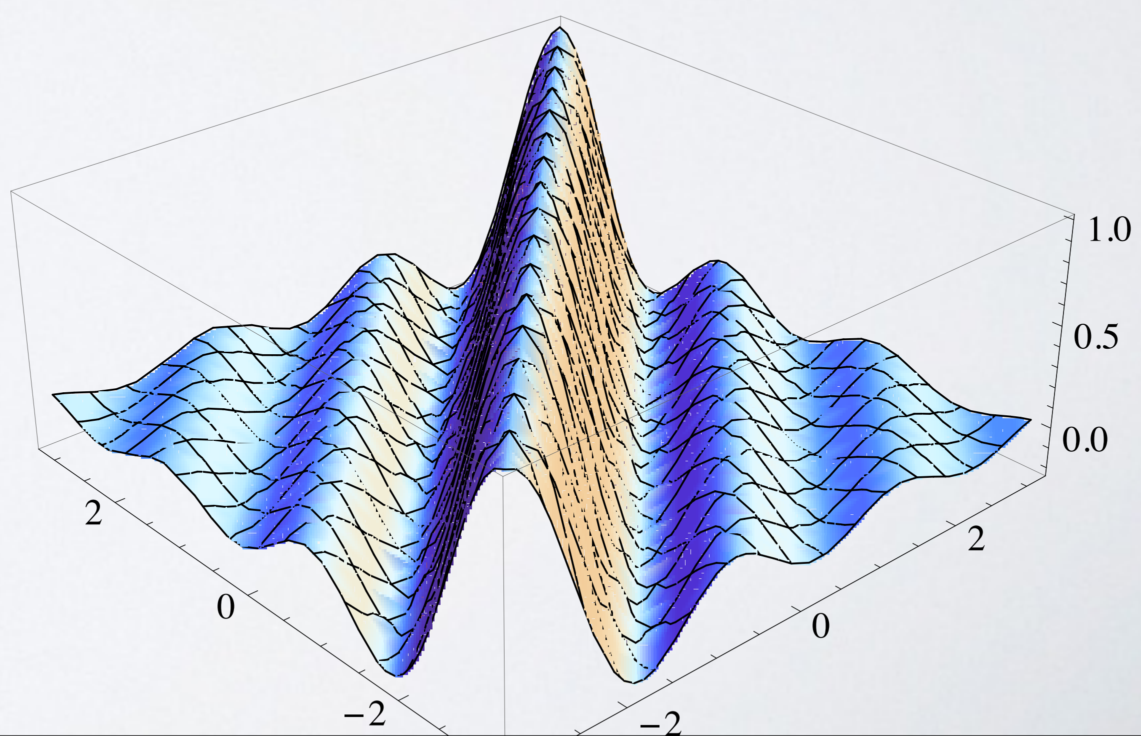
$$\lim_{N\to\infty} \frac{1}{\rho_{SC}(x)\sqrt{N}} K_N\left(\sqrt{N}x + \frac{y_1}{\sqrt{N}\rho_{SC}(x)}, \sqrt{N}x + \frac{y_2}{\sqrt{N}\rho_{SC}(x)}\right)$$
$$= K_{\sin}(y_1, y_2), \quad |x| \leq \sqrt{2},$$

with

$$K_{\sin}(y_1, y_2) = \frac{\sin(\pi(y_1 - y_2))}{\pi(y_1 - y_2)}.$$

Sine kernel



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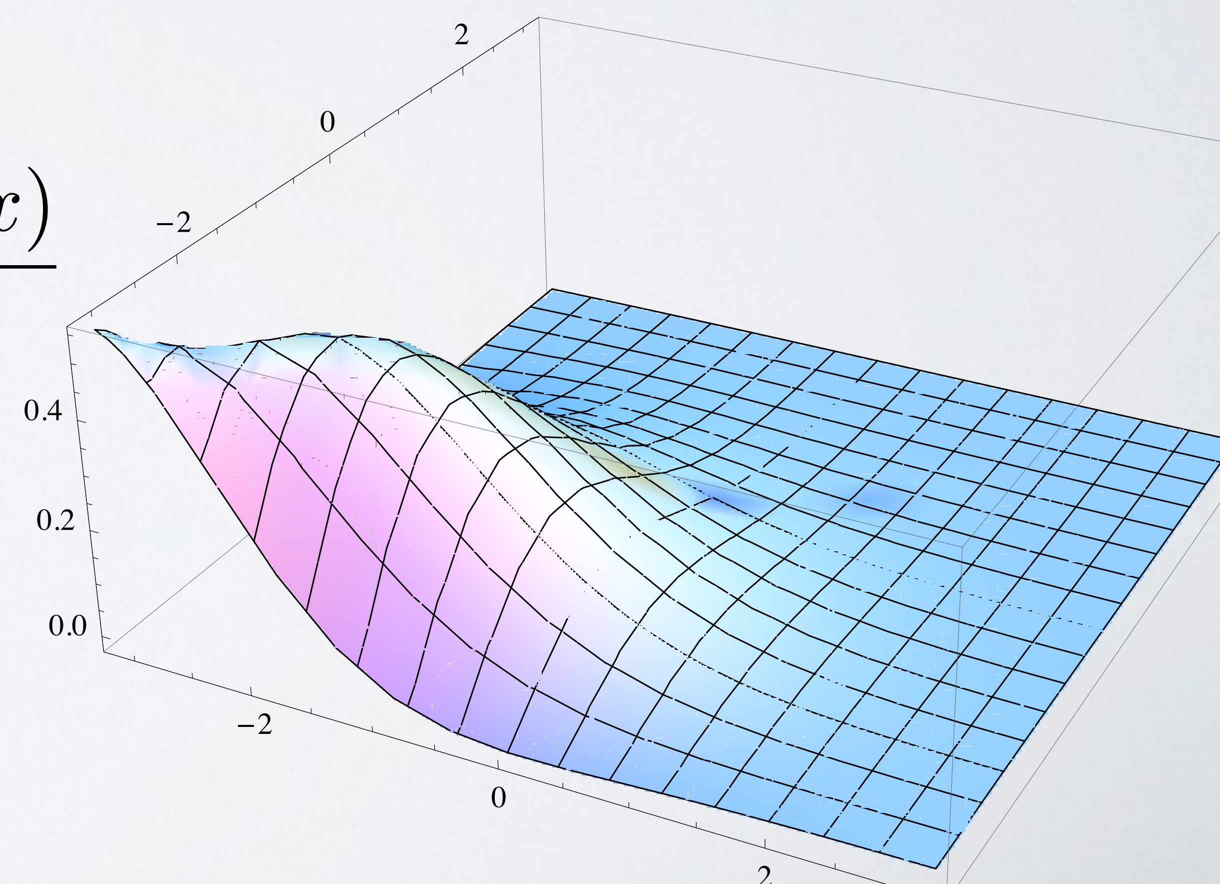
$$\lim_{N\to\infty} N^{1/6} K_N\left(\sqrt{2N} + \frac{x}{N^{1/6}}, \sqrt{2N} + \frac{y}{N^{1/6}}\right) = K_{Ai}(x,y), \quad x, y \in \mathbb{R},$$

with

$$K_{Ai}(x,y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y}$$

where Ai is the Airy function.

Airy kernel



CONNECTIONS WITH O. P.

Both $K_{\sin}$ and K_{Ai} appear (in the bulk and at the "soft" edge of the spectrum) for much more general weights w.

Universality!

Also Multiple Orthogonal Polynomials?

CONNECTIONS WITH M.O. P.

Recall that given a system of measures $\mu_1, \ldots, \mu_s$ and a vector $\boldsymbol{n} = (n_1, \ldots, n_s) \subset \mathbb{N}^s$, the multiple orthogonal polynomials (MOP) of type II, P_n, are polynomials of degree $\leq |\boldsymbol{n}|$ such that

$$\int P_n(x) x^k \, d\mu_j(x) = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, \ldots, s.$$

(There are also polynomials of Type I, not so important here).

Their asymptotics is also described by an equilibrium problem for logarithmic potentials, but much more involved.

Instead of one measure, we now have a vector of measures that interact with each other according to a certain law (given by the geometry of the problem).

We seek for a minimizer of the global energy, that takes into account all the components.

This is an example of the vector equilibrium.

RANDOM MATRICES WITH EXTERNAL SOURCE

Now the probability distribution on $N \times N$ Hermitian matrices is

$$\mathcal{P}_N(M)\,dM = \frac{1}{Z_N} e^{-Tr(V(M)) - AM} dM,$$

where A is a **fixed** $N \times N$ Hermitian matrix.

The eigenvalues form again a determinantal process (Mehta).

Consider the average characteristic polynomial,

$$P_N(z) = \mathbb{E}\left[\det\left(zI - M\right)\right]$$

I'm a **multiple** OP (of Type II)

for such an ensemble. Then

$$\int_{\mathbb{R}} P_N(x) x^k e^{-V(x) - a_j x} dx = 0, \quad k = 0, 1, \ldots, m_j - 1,$$

where a_j is the j-th eigenvalue of A of multiplicity m_j.

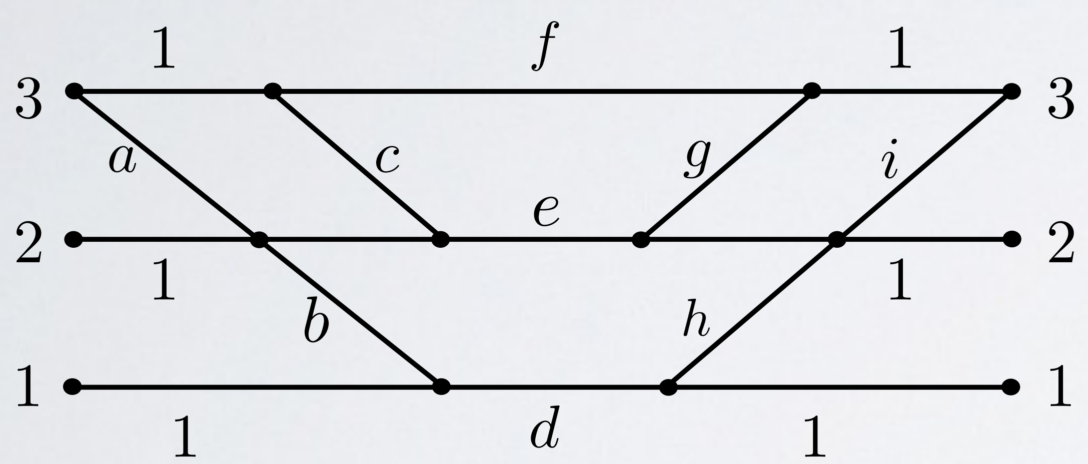
Let’s talk about random paths

PLANAR NETWORKS

Planar network (Γ, ω) = acyclic directed $(\rightarrow)$ planar graph Γ with scalar weights ω assigned to its edges.

Weight of a directed path from i to $j = \prod$ weights of its edges.

Weight matrix $= n \times n$ matrix (a_{ij}), with $a_{ij} = \sum$ weights of all paths $i \rightarrow j$.



$$\begin{pmatrix} d & dh & dhi \\ bd & bdh + e & bdhi + eg + ei \\ abd & abdh + ae + ce & abdhi + (a+c)e(g+i) + f \end{pmatrix}$$

Weight of a collection of directed paths $= \prod$ of their weights.

Lindström Lemma: *a minor given by the intersection of rows I and columns J of the weight matrix of a planar network = the sum of weights of all collections of vertex-disjoint paths $I \rightarrow J$.*

Example: minor $(2,3) \times (2,3)$ is equal to $bcdegh + bdfh + fe$.

NON-INTERSECTING RANDOM PATHS

Assume we have a 1-D strong Markov process with continuous sample paths (1-D diffusion process) with the transition probability density function $p_t(x, y)$.

Example: Brownian motion

NON-INTERSECTING RANDOM PATHS

Assume we have a 1-D strong Markov process with continuous sample paths (1-D diffusion process) with the transition probability density function $p_t(x, y)$.

Karlin & McGregor, 1959]: *Suppose that n labelled particles start out in states $a_1 < \cdots < a_n$ and execute the process simultaneously and independently. Then the determinant*

$$\det \begin{pmatrix} p_t(a_1, b_1) & \dots & p_t(a_1, b_n) \\ \vdots & & \vdots \\ p_t(a_n, b_1) & \dots & p_t(a_n, b_n) \end{pmatrix}$$

is equal to the probability that at time t the particles will be found in states $b_1 < \cdots < b_n$ respectively without any two of them ever having been coincident (simultaneously in the same state) in the intervening time.

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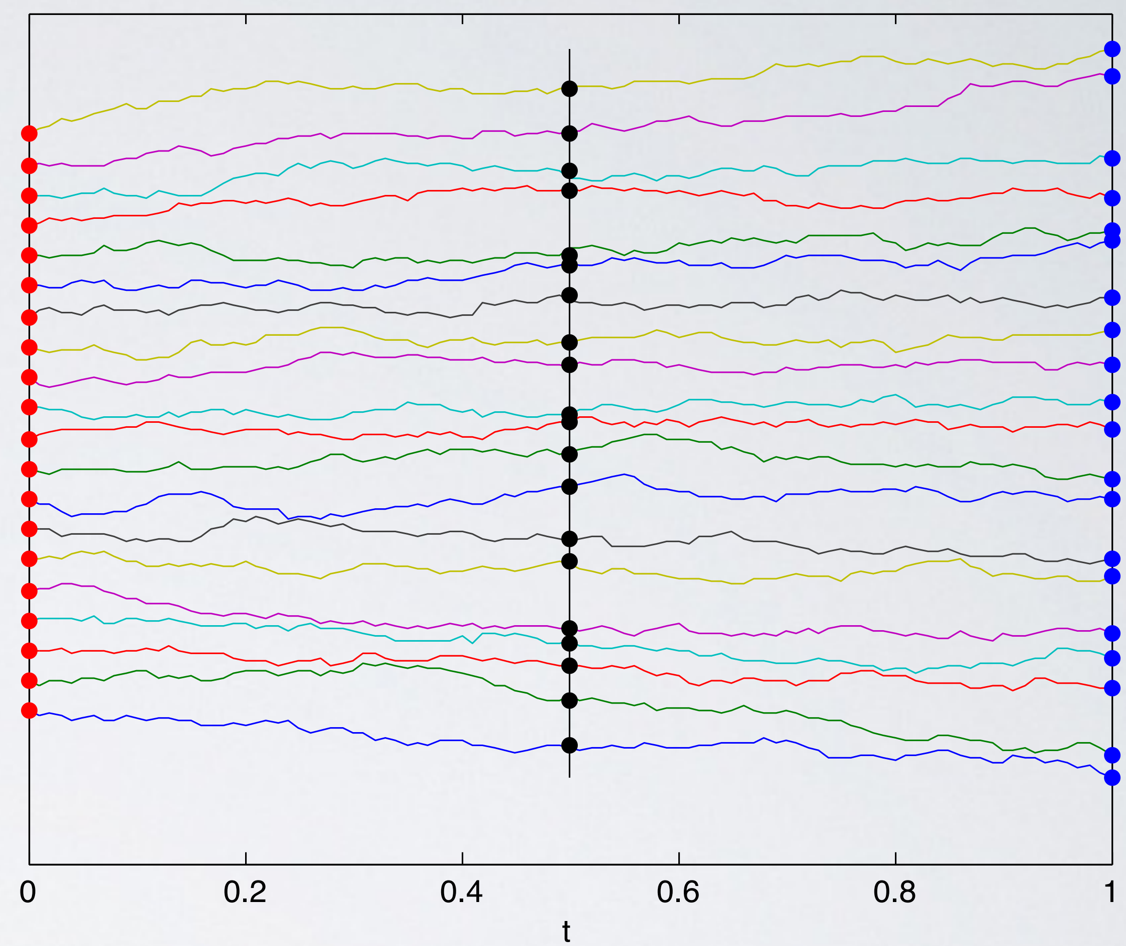
is eq… at time t the particles will be found … without any two of them ever … usly in the same state) in the intervening …

A determinantal process!

NON-INTERSECTING RANDOM PATHS

Assume we are given



- initial positions at time 0: $a_1 < a_2 < \cdots < a_n$
- end positions at time $T > 0$: $b_1 < b_2 < \cdots < b_n$

Then the (conditional) joint probability density function on $\mathbb{R}^n$ of the n independent instances of this diffusion process with non-intersecting paths at an intermediate time $0 < t < T$ has the form

$$\mathcal{P}(x_1, \ldots, x_n) = \frac{1}{Z_n} \det(p_t(a_i, x_j)) \det(p_{T-t}(x_i, b_j)).$$

NON-INTERSECTING RANDOM PATHS

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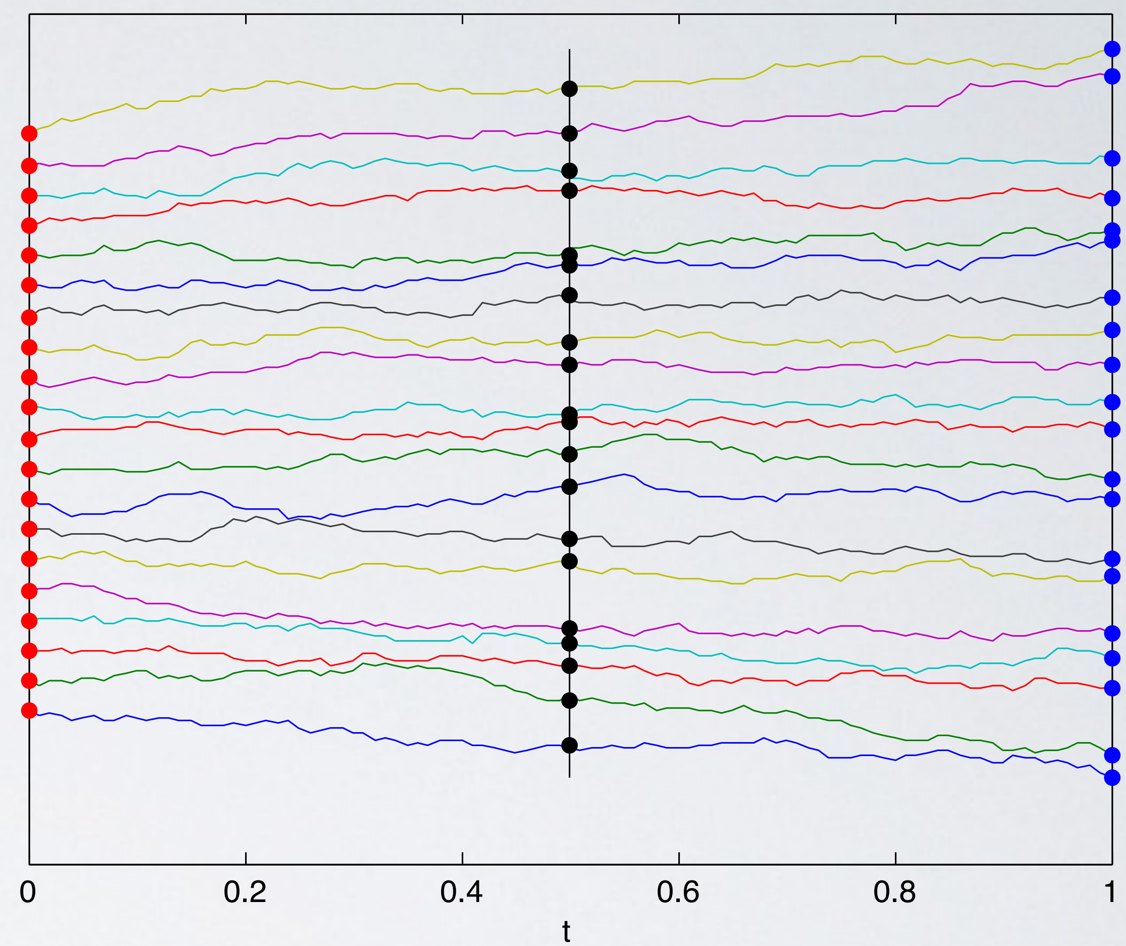


Again, the correlation kernel K_n, expressed in terms of $p_t(a_j, x)$ and $p_{T-t}(x, b_j)$, can be rewritten as

$$K_n(x, y, t) = \sum_{j=1}^{n} \phi_j(x,t)\psi_j(y,t) \qquad \text{where } \phi_i \perp \psi_j.$$

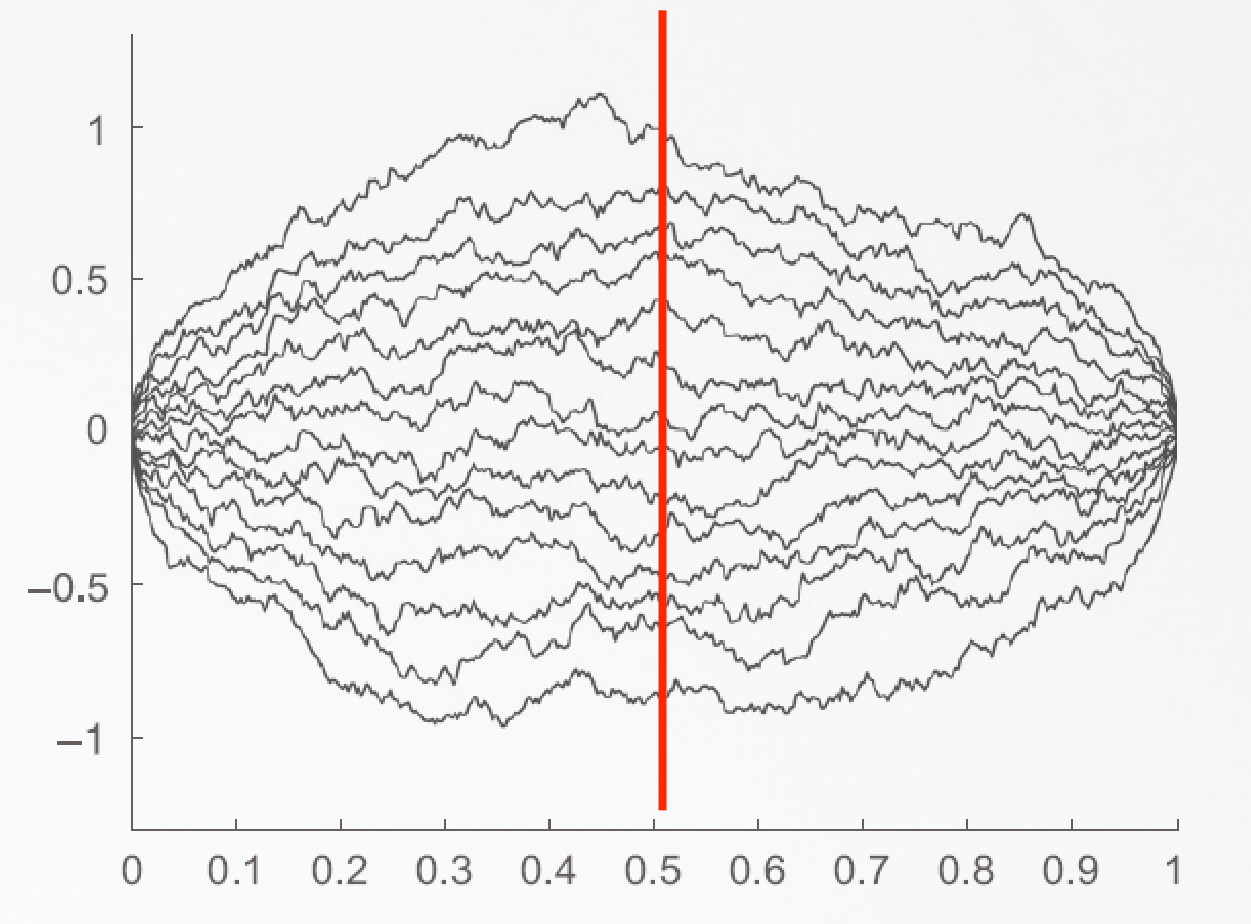
We are interested in the confluent case (some a_j or b_j may collide).

NON-INTERSECTING RANDOM PATHS

Consider the Brownian motion:

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\, e^{-(x-y)^2/(2t)}$$

Assume all $a_j = 0$ and $b_j = 0$



Let us see what happens in the Karlin & McGregor determinant, where we have rows of the form

$$\begin{pmatrix} p_t(a_1,b_1), & \dots & p_t(a_1,b_N) \\ p_t(a_2,b_1), & \dots & p_t(a_2,b_N) \\ \dots & \dots & \dots \end{pmatrix}$$

when $a_2 \to a_1$.

NON-INTERSECTING RANDOM PATHS

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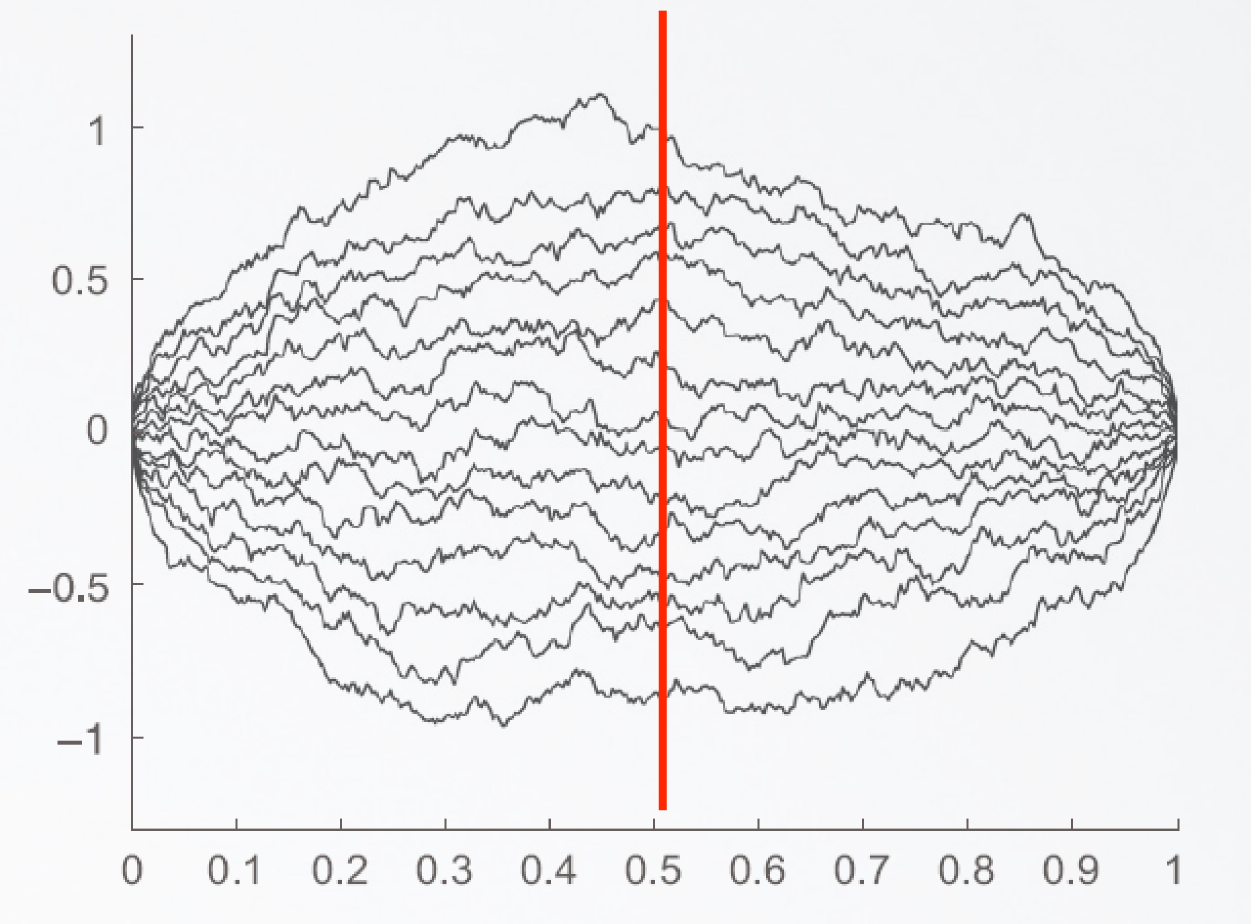

Wrong: $$\begin{pmatrix} p_t(a_1,b_1), & \dots & p_t(a_1,b_N) \\ p_t(a_1,b_1), & \dots & p_t(a_1,b_N) \\ \dots & \dots & \dots \end{pmatrix}$$

Right: $$\begin{pmatrix} p_t(a_1,b_1), & \dots & p_t(a_1,b_N) \\ p'_t(a_1,b_1), & \dots & p'_t(a_1,b_N) \\ \dots & \dots & \dots \end{pmatrix}, \text{ where } p'_t(a,b) = \frac{d}{da} p_t(a,b).$$

NON-INTERSECTING RANDOM PATHS

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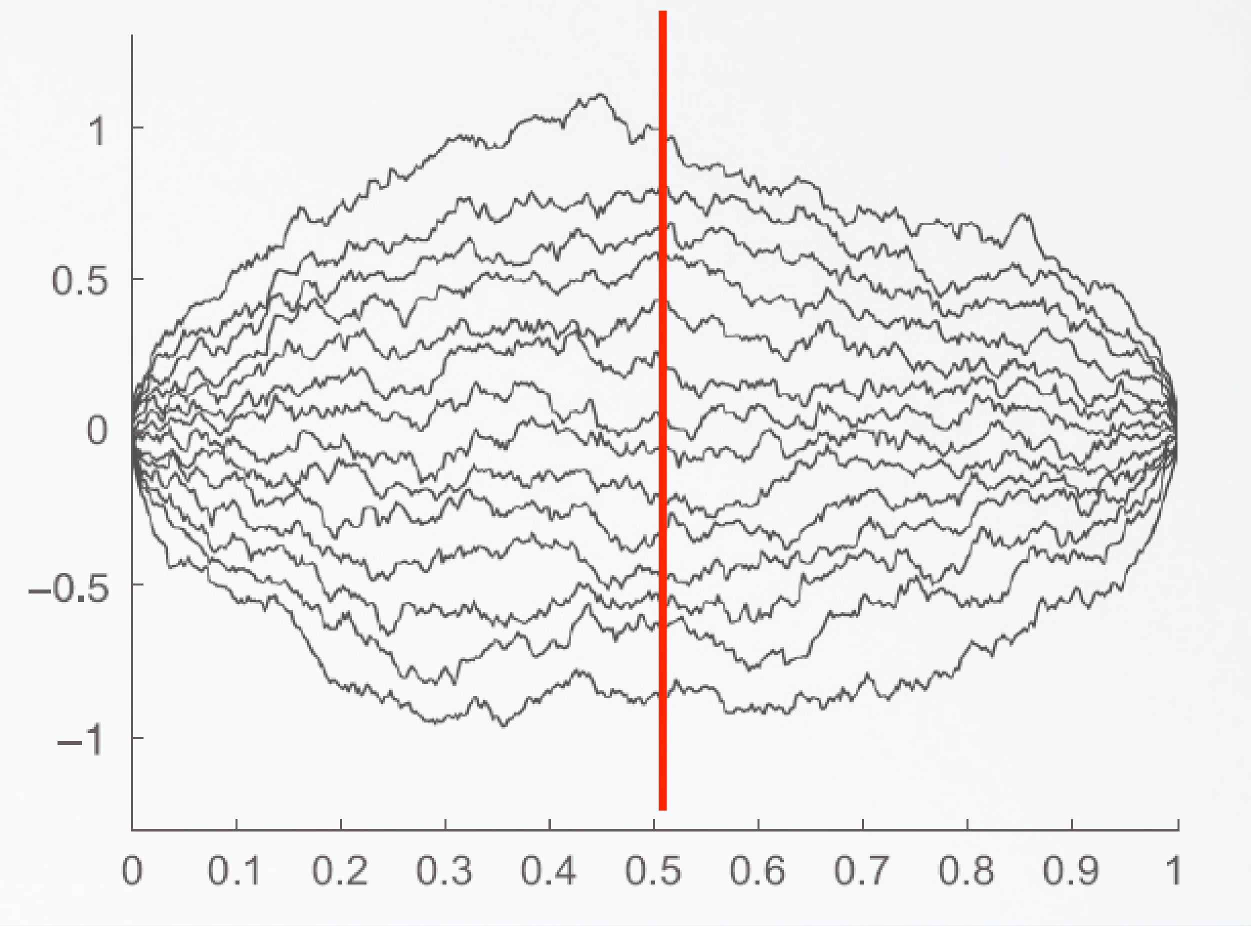

In the expression for K_n we may take $\phi_j(x) = \psi_j(x) = \pi_j(x)e^{-x^2/c(t)}$, where $\pi_j = j$th Hermite polynomial.

Conclusion: the correlation kernel K_n for the confluent non-intersecting paths performing a Brownian motion coincides with that for the spectrum of the GUE. Both processes are statistically identical!

NON-INTERSECTING RANDOM PATHS

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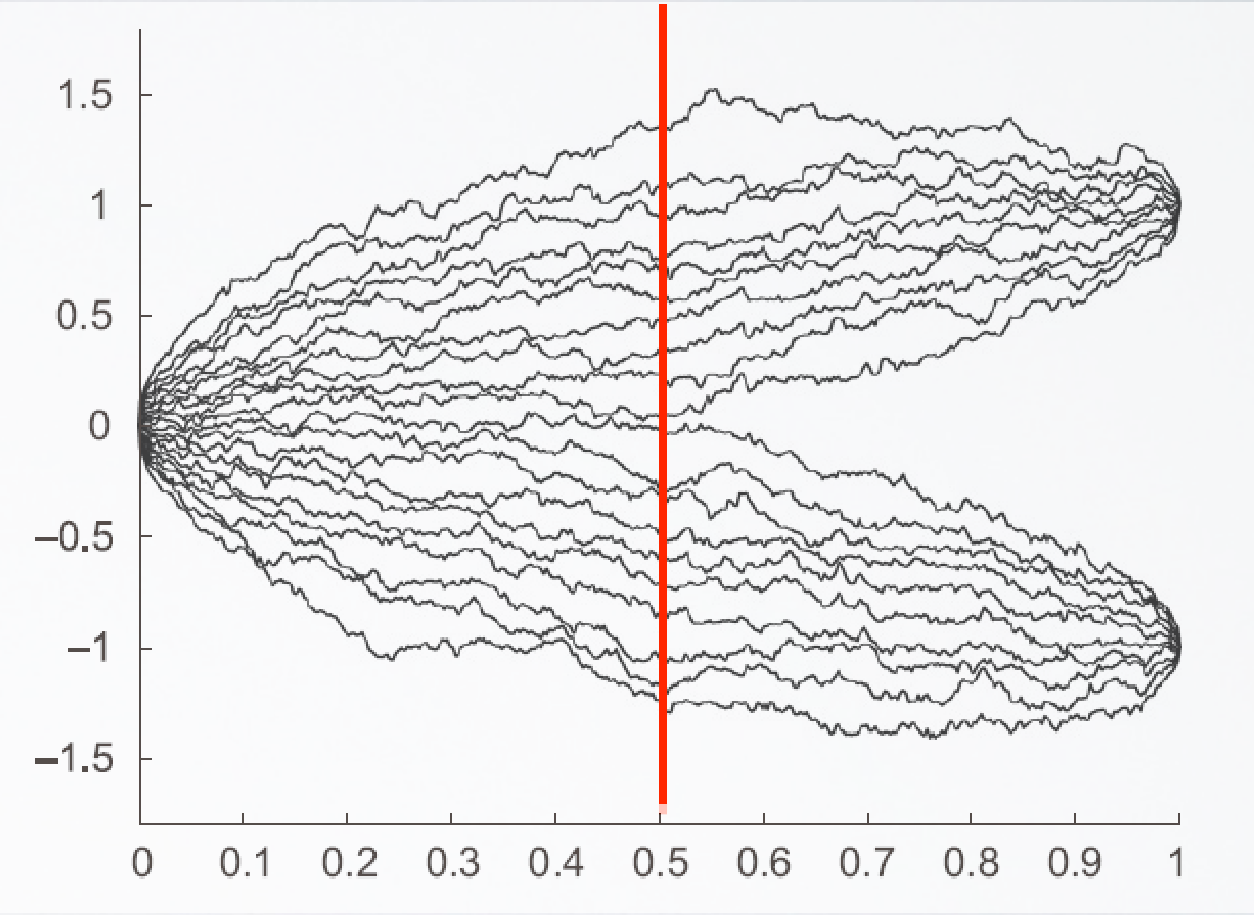


Assume all $a_j = 0$ but half $b_j = b^{(1)}$ and other half $b_j = b^{(2)}$, $b^{(1)} \neq b^{(2)}$:

Now the correlation kernel is built out of multiple Hermite polynomials, with

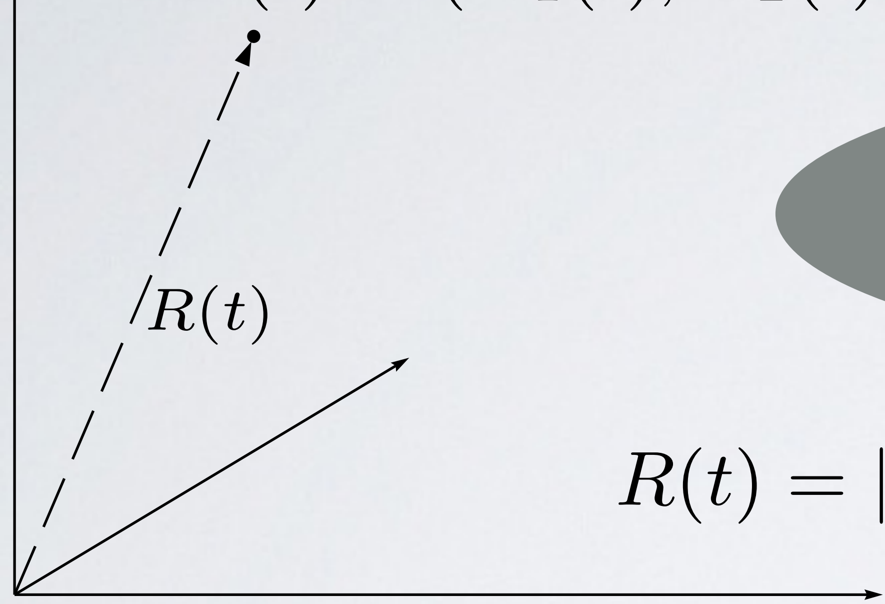
$$w_1(x) = e^{-\frac{(x-b^{(1)})^2}{2t}}, \quad w_1(x) = e^{-\frac{(x-b^{(2)})^2}{2t}} \quad \text{(multiple OP ensemble).}$$

Statistically = the eigenvalues of the GUE with external source, when matrix A has only 2 eigenvalues: $b^{(1)}$ and $b^{(1)}$.

That was a dirty trick to create Multiple Orthogonal Polynomials! Can they appear naturally?

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

$$\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t))$$



A Brownian motion each coordinate

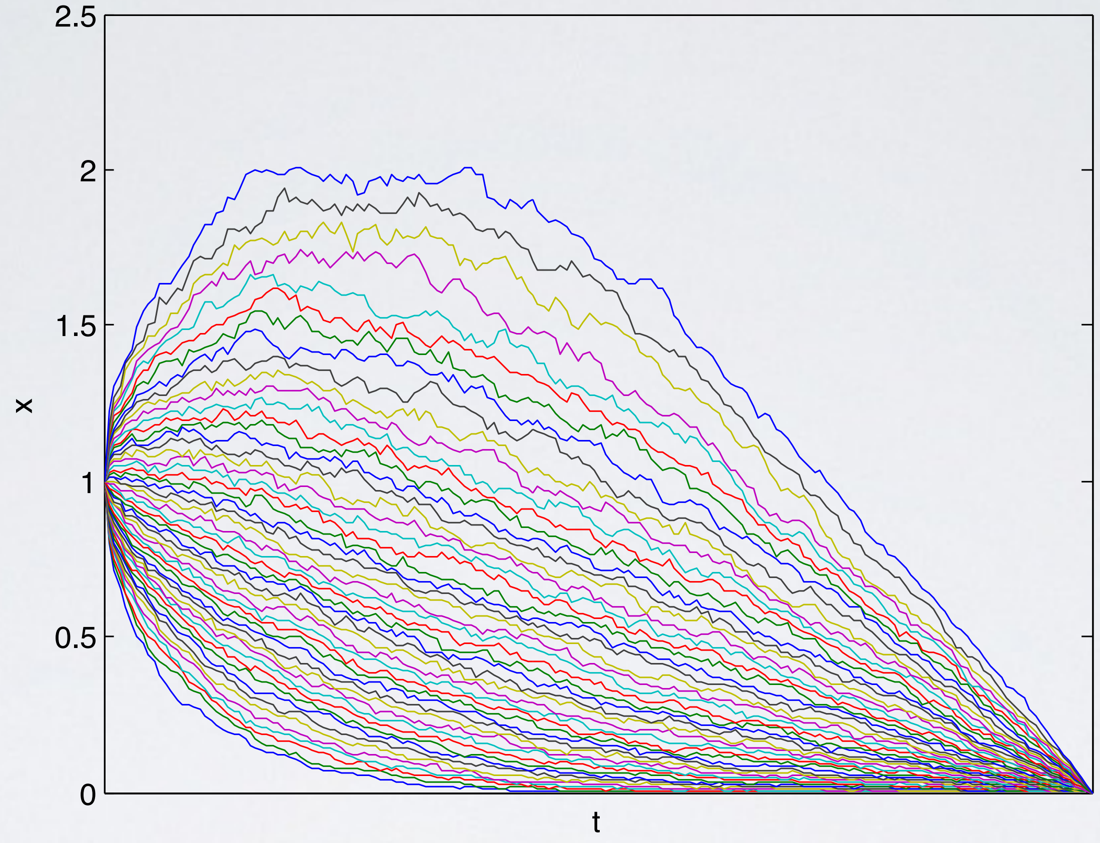
$$R(t) = \|\mathbf{X}(t)\|_2 = \sqrt{X_1(t)^2 + X_2(t)^2 + X_3(t)^2}, \quad t \geq 0,$$

is the Bessel process with parameter $\alpha = \frac{d}{2} - 1$, while $R^2(t)$ is the squared Bessel process BESQ^d. Its transition probability is

$$p_t^{\alpha}(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\alpha/2} e^{-(x+y)/(2t)} I_\alpha\left(\frac{\sqrt{xy}}{t}\right), \qquad x, y > 0,$$

$$p_t^{\alpha}(0,y) = \frac{y^\alpha}{(2t)^{\alpha+1}\Gamma(\alpha+1)} e^{-y/(2t)}, \qquad y > 0.$$

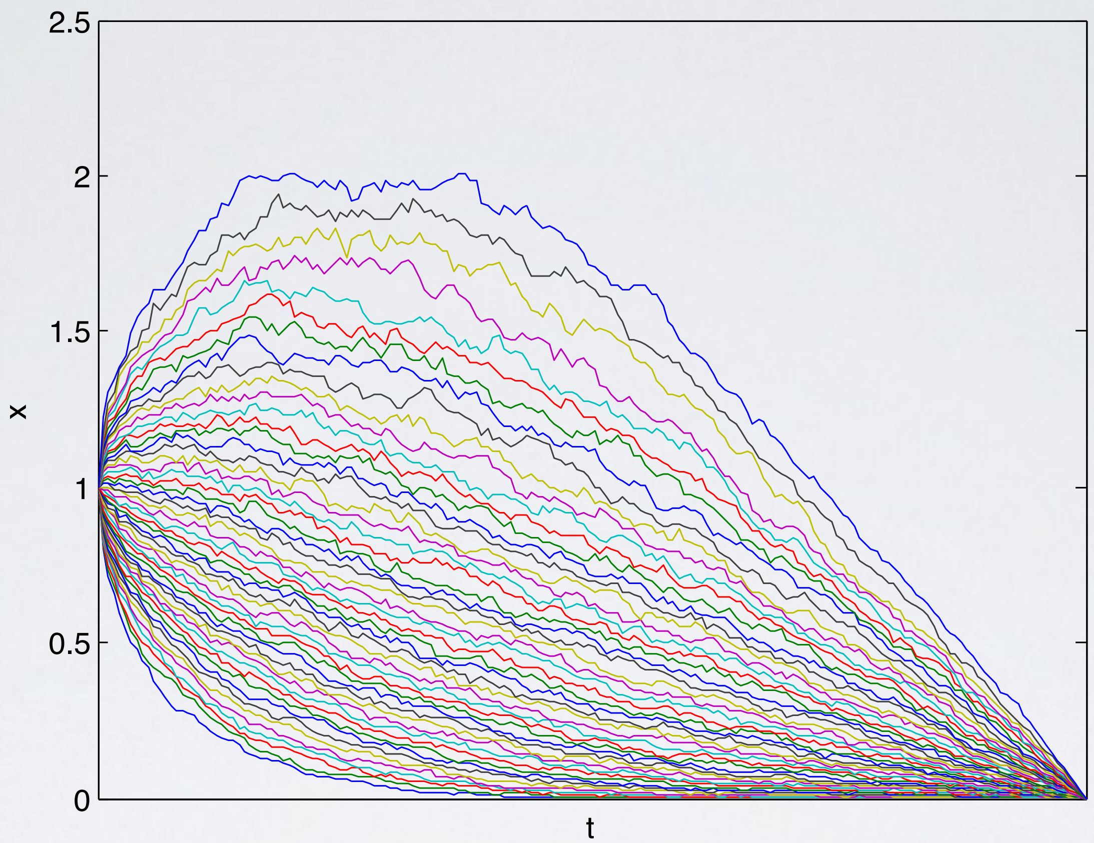
SQUARED BESSEL PROCESS IN D=3 (BESQ^3)



Consider a system of n particles performing BESQ^d and conditioned never to collide with each other, in the confluent case where all particles start ($t = 0$) at the same value $a > 0$ and all end ($t = T$) at 0.

Studied by König–O'Connell (2001), Katori-Tanemura (2007), and Desrosiers-Forrester (2008). Related work by Tracy-Widom (2007) on non-intersecting Brownian excursions.

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)



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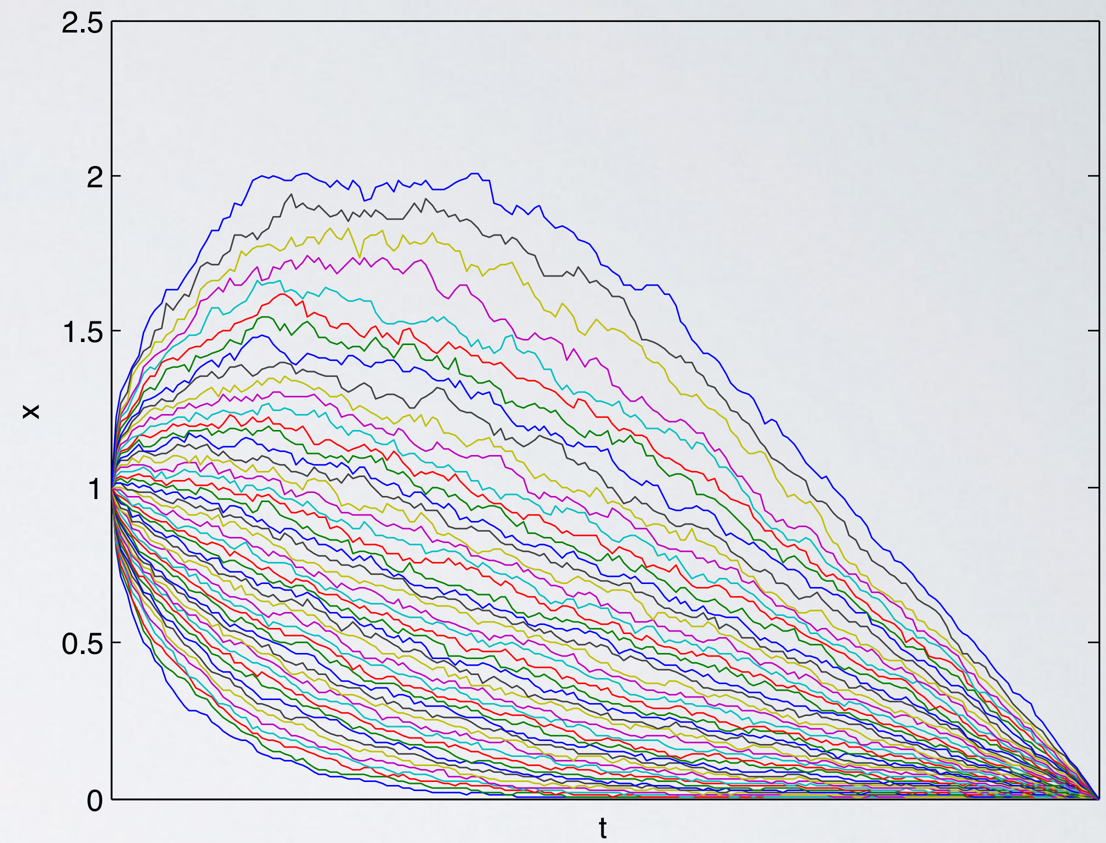
We are interested in the limit $n \to \infty$, under time rescaling ("slow motion"):

$$t \mapsto \frac{t}{2n}, \quad T \mapsto \frac{1}{2n} \quad \Rightarrow \quad 0 < t < 1.$$

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

These non-intersecting squared Bessel paths constitute a multiple orthogonal ensemble!



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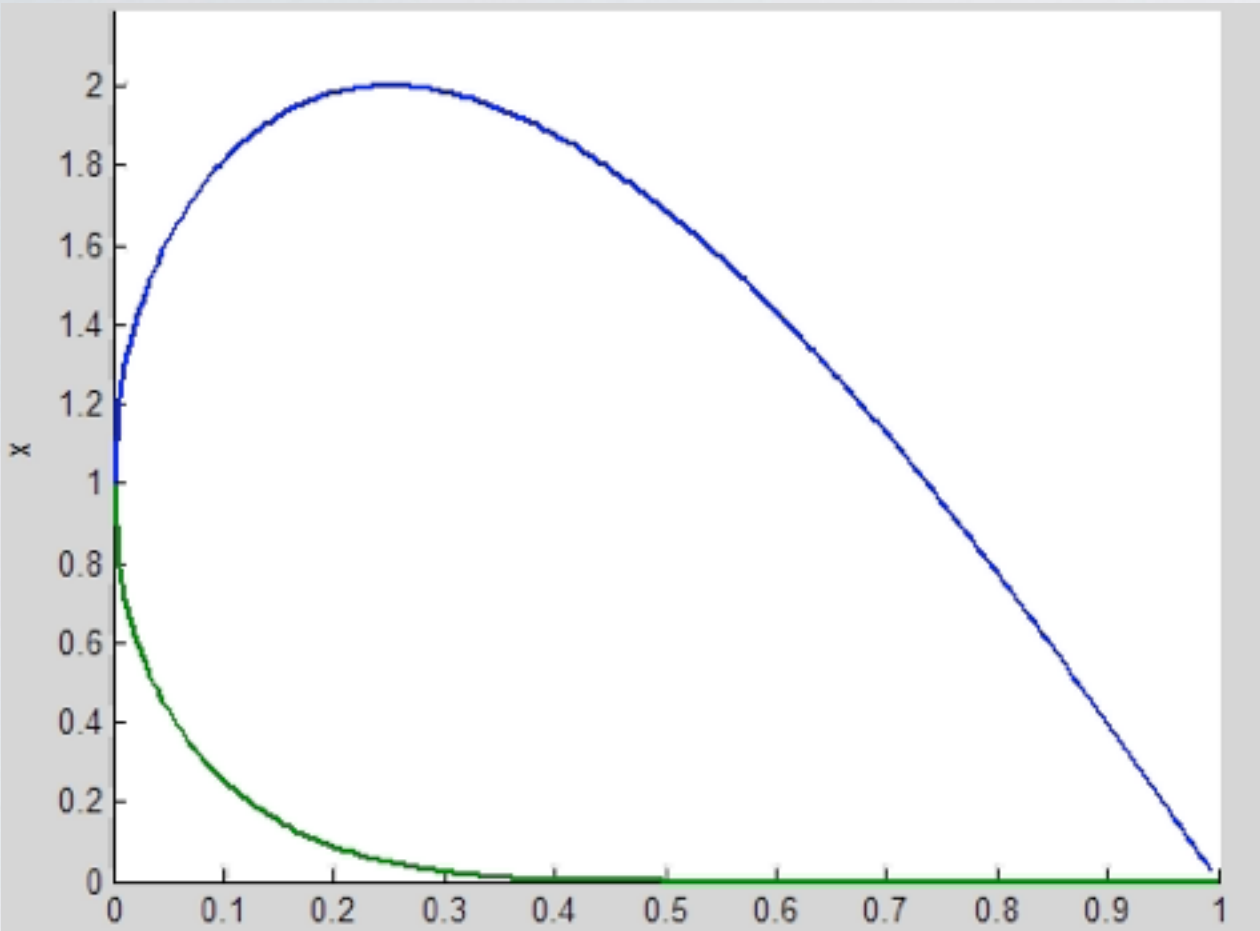
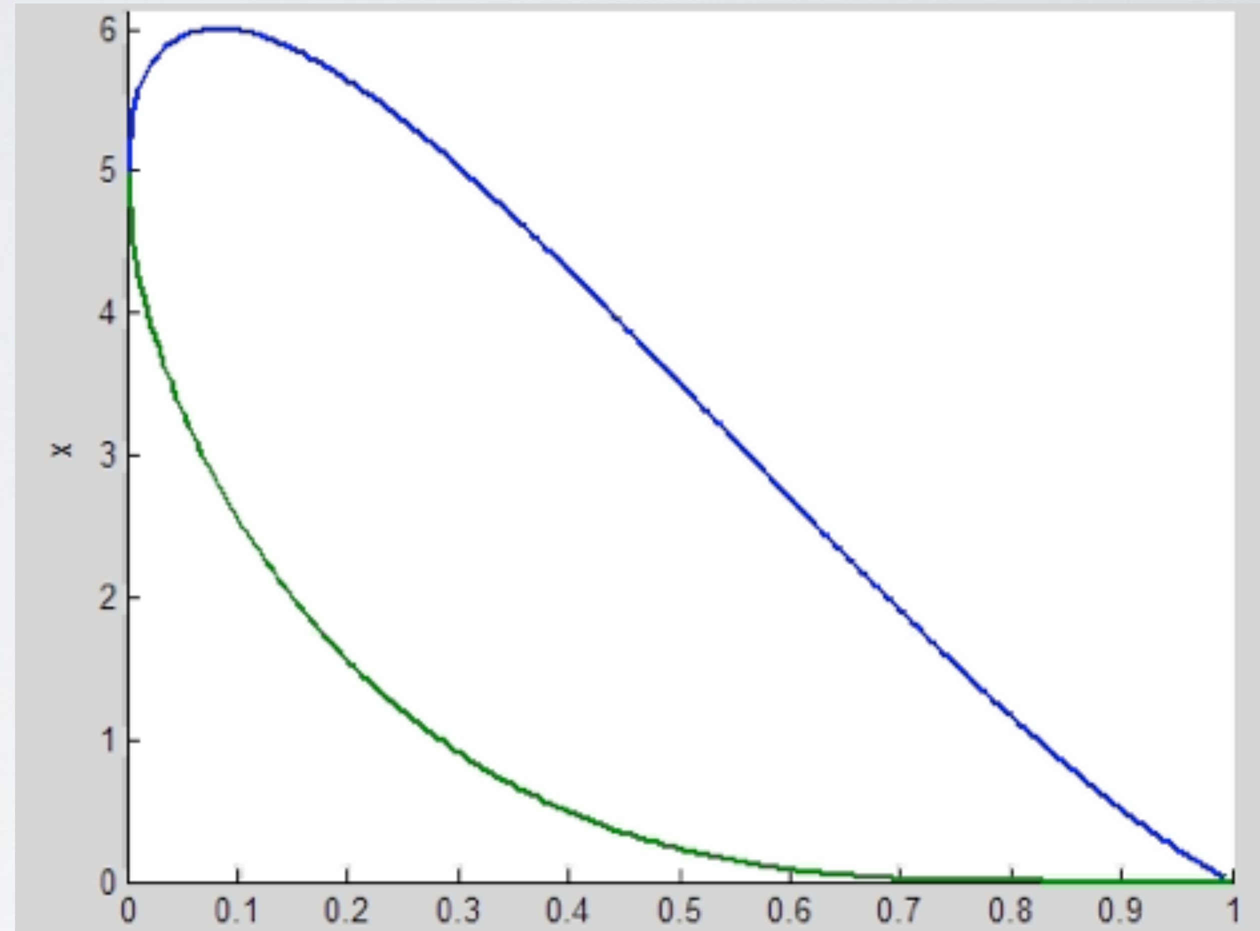
SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:





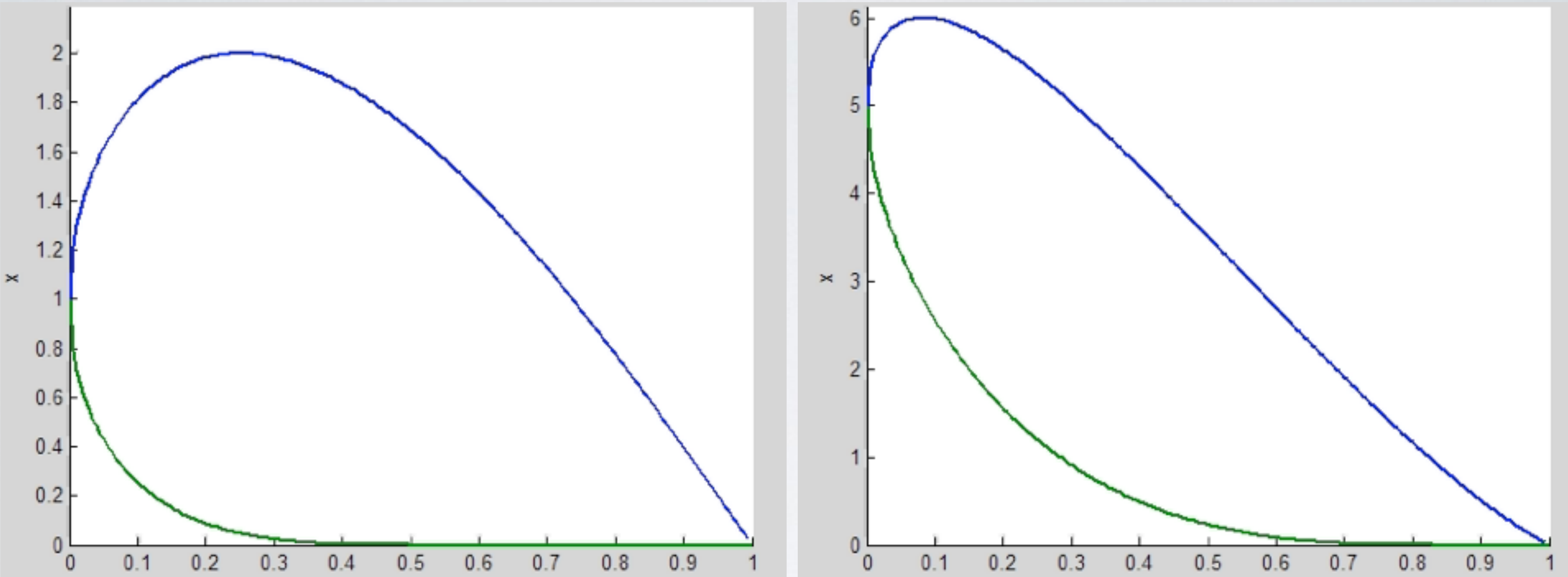
$\forall t \in (0,1)$, the limiting mean density of the positions of the paths

$$\textcolor{red}{\text{(explicit)}}\ \rho(x) = \rho(x;t) = \lim_{n\to\infty} \frac{1}{n} K_n(x,x;t)$$

exists, and is supported on $[p(t), q(t)]$, where $x = p(t)$, $x = q(t)$ are non-negative roots of an explicit polynomial of degree 3.

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

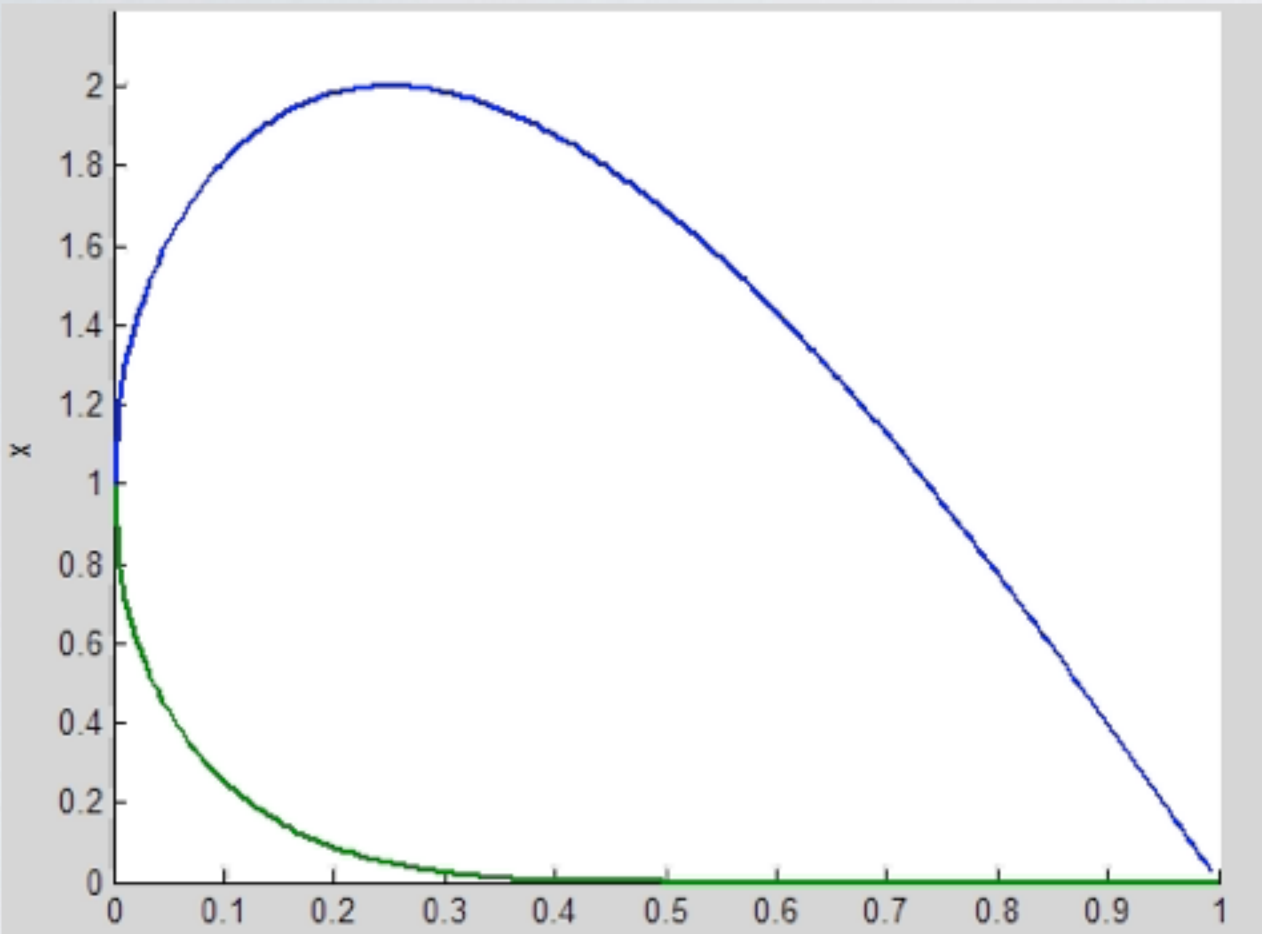
Behavior for large n:

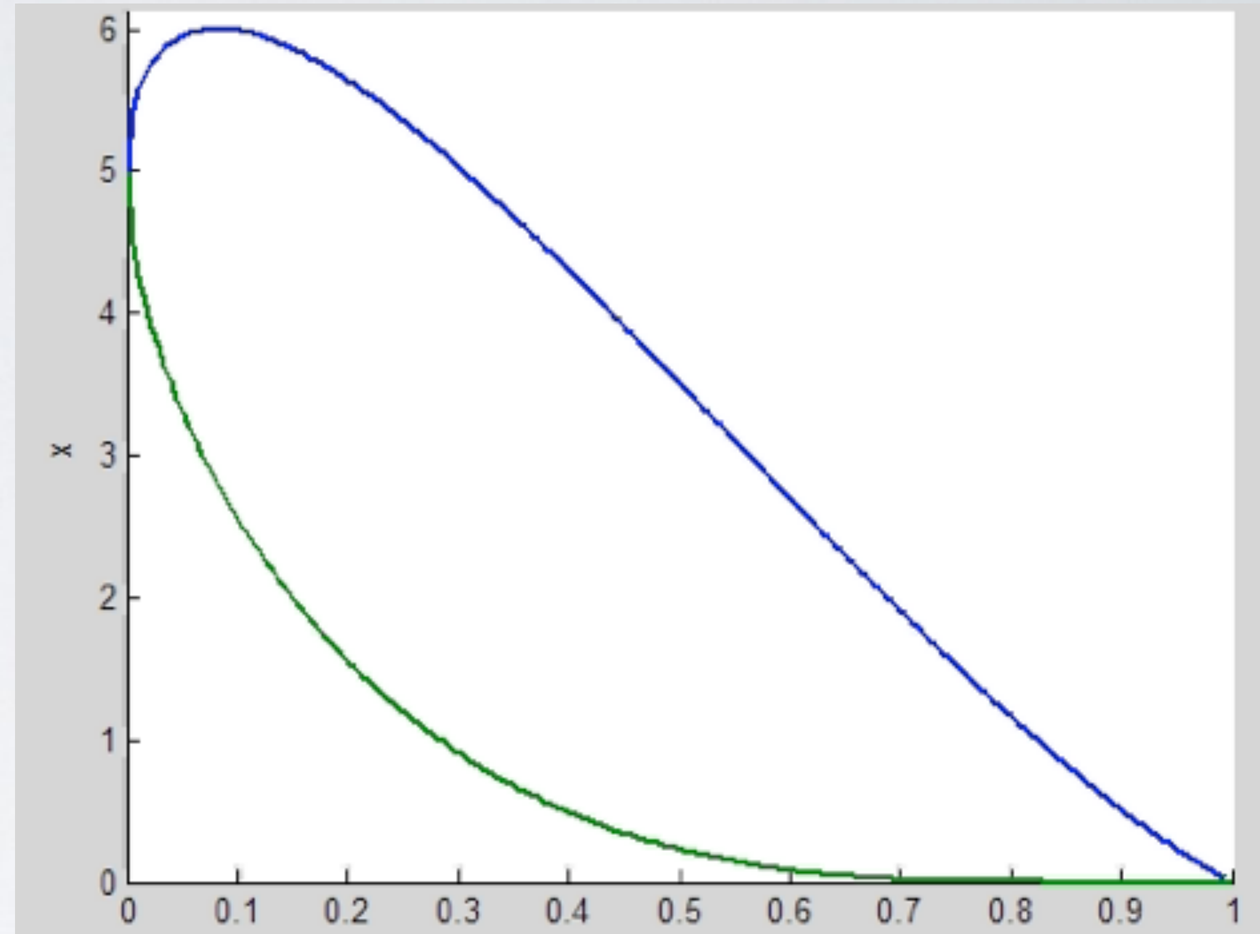


The **lower** boundary curve $x = p(t)$ is positive for $t < t^* = a/(a+1)$ and it is zero for $t \geq t^*$. At $t = t^*$ it has continuous first and second order derivatives.

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:





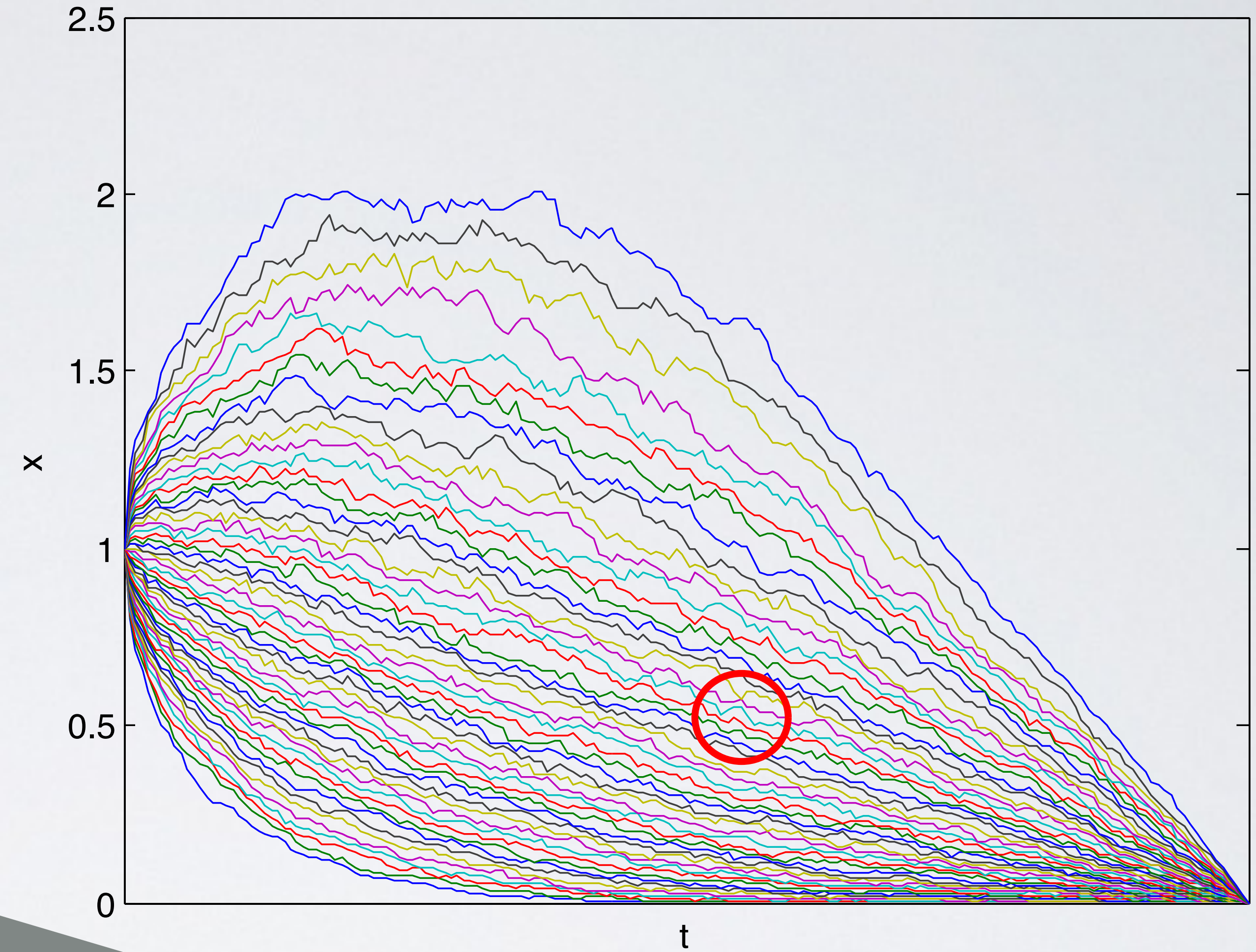
The upper boundary curve $x = q(t)$ has a slope $q'(1) = -4$ at $t = 1$ which is independent of the value of a.

It is concave if $a \leq 1$, and not concave on $[0, 1]$ if $a > 1$.

The maximum of the upper boundary curve $x = q(t)$ is $a + 1$.

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:



In the bulk:

Sine kernel

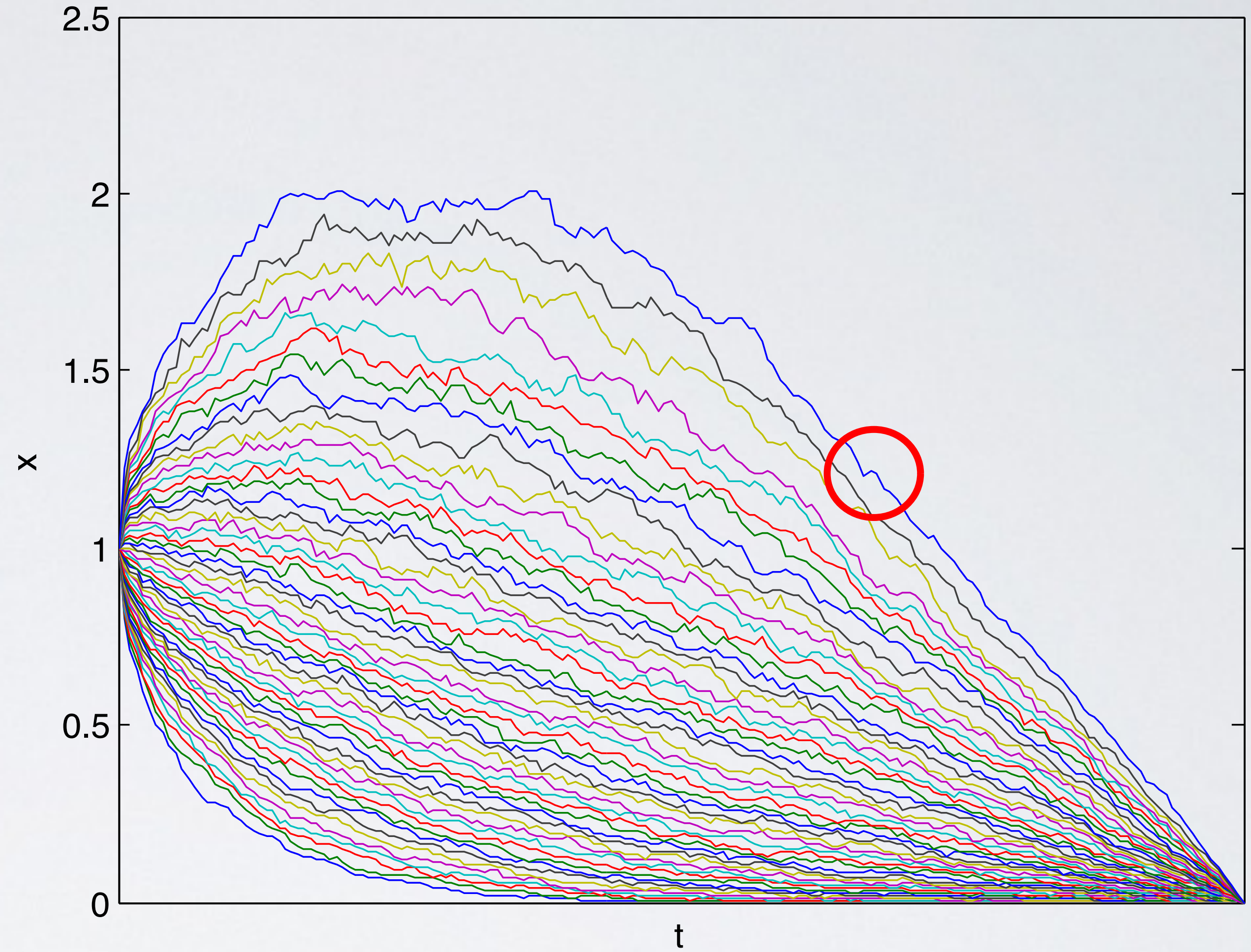
If $t \neq t^*$, then for $x^* \in (p(t), q(t))$, we have

$$\lim_{n\to\infty} \frac{1}{n\rho(x^*)} K_n\left(x^* + \frac{x}{n\rho(x^*)}, x^* + \frac{y}{n\rho(x^*)}, t\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

uniformly for x and y in compact subsets of $\mathbb{R}$.

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:



At the soft edge:

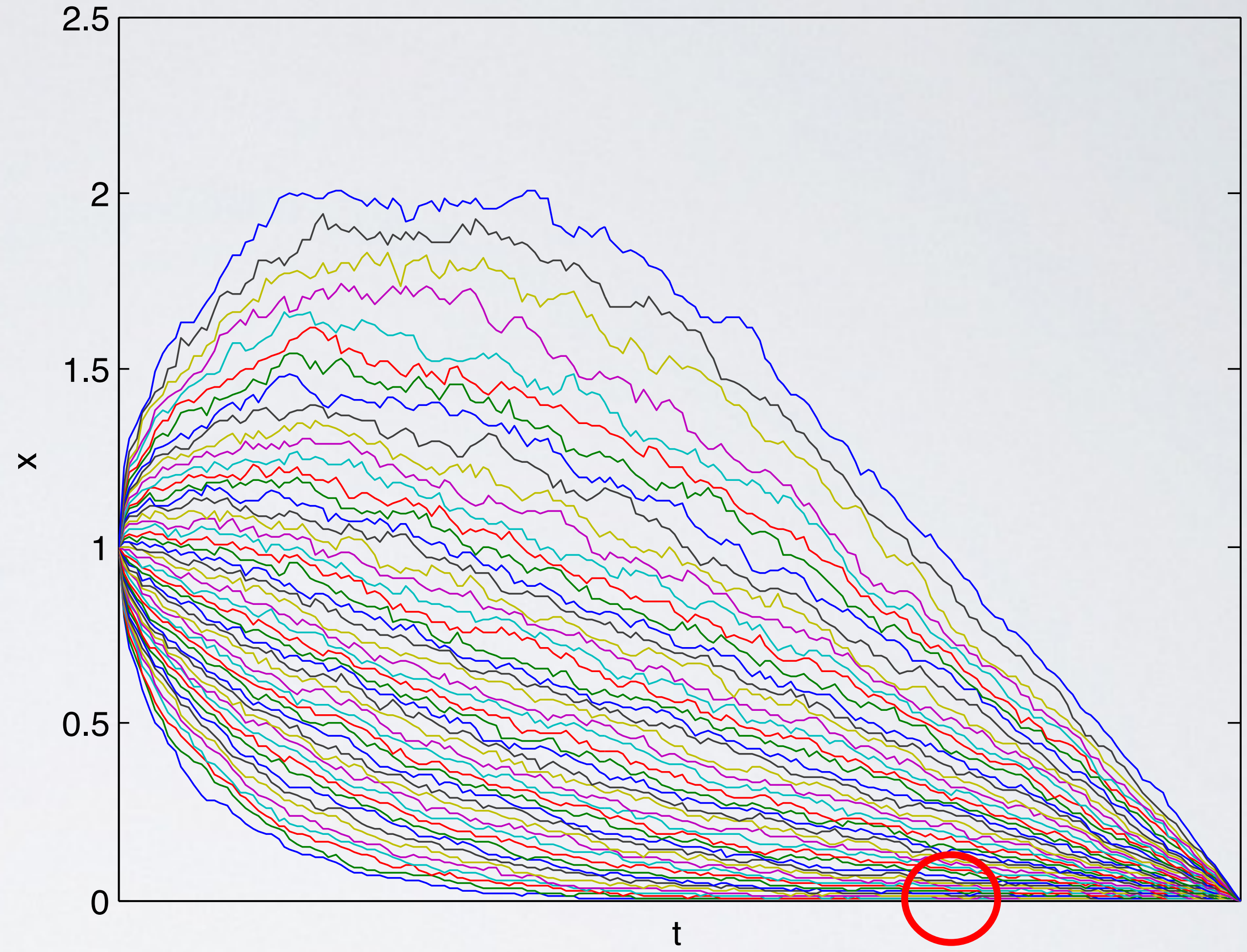
Airy kernel

If $t \neq t^*$, then for some constant $c > 0$, and $x, y > 0$,

$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_n\left(q(t) + \frac{x}{cn^{2/3}}, q(t) + \frac{y}{cn^{2/3}}, t\right)$$
$$= \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}.$$

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:



At the hard edge:

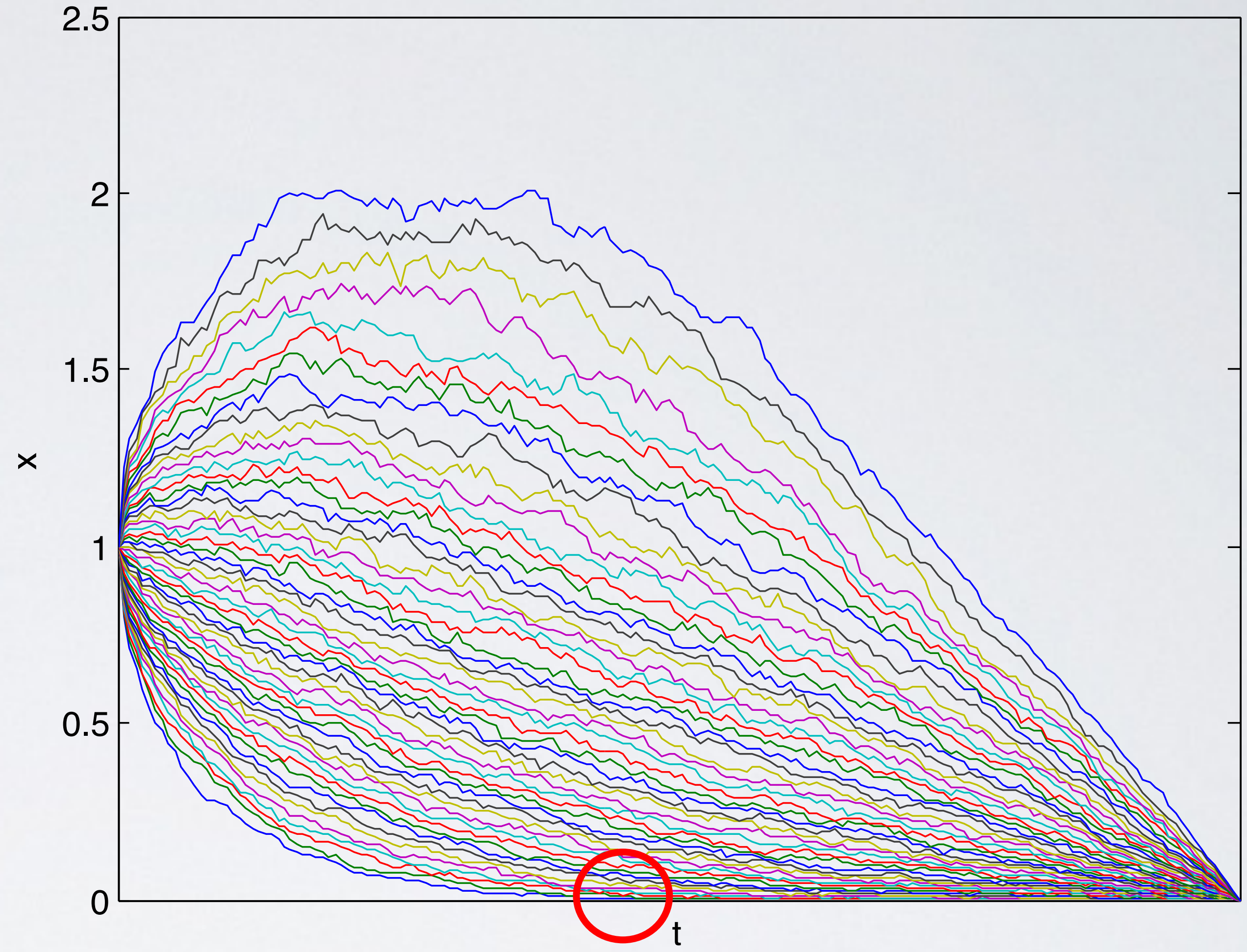
Bessel kernel

If $t > t^*$, then for some constant $c > 0$, and $x, y > 0$,

$$\lim_{n\to\infty} \frac{1}{cn^2} K_n\left(\frac{x}{cn^2}, \frac{y}{cn^2}, t\right) = \left(\frac{y}{x}\right)^{\alpha/2} \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}.$$

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:



Double scaling limit at $x^* = 0$ and $t = t^*$:

A new one

For some $c > 0$ and $x, y > 0$,

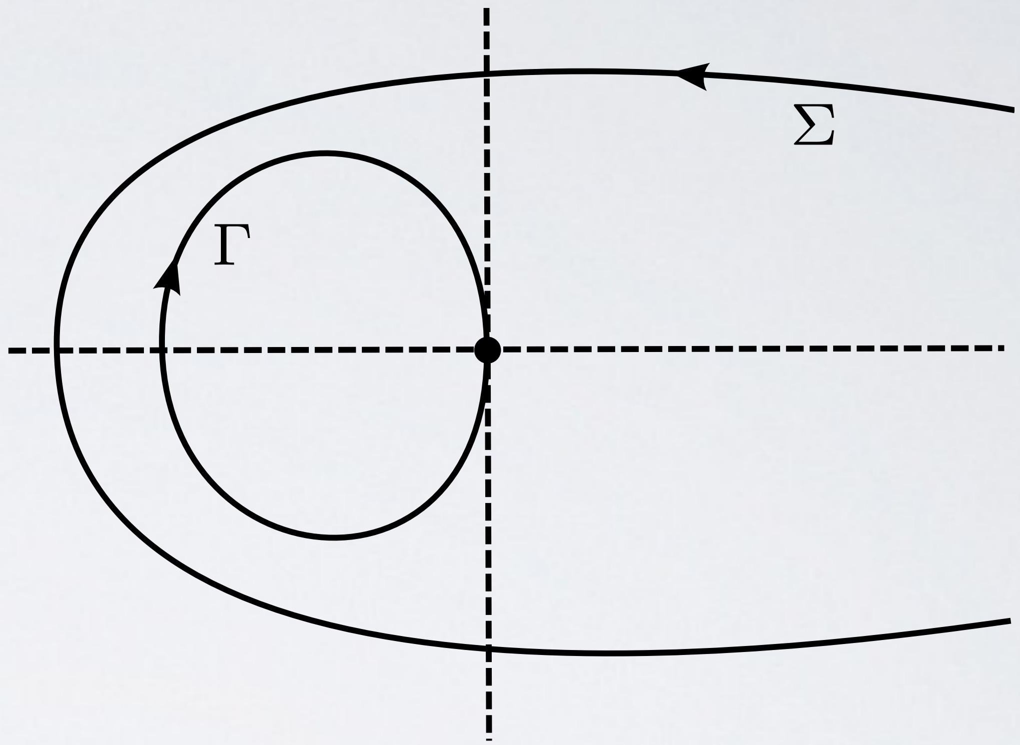
$$\lim_{n\to\infty} \frac{c}{n^{3/2}} K_n\left(\frac{cx}{n^{3/2}}, \frac{cy}{n^{3/2}}, t^* - \frac{c\tau}{\sqrt{n}}\right) = K^{crit}(x, y, \tau),$$

a new kernel involving a solution of $xy''' + (\alpha+2)y'' - \tau y' - y = 0$.

SQUARED BESSEL PROCESS IN D=3 (BESQ^3)

Behavior for large n:

Double scaling limit at $x^* = 0$ and $t = t^*$:



$$K^{crit}(x, y, \tau) = \frac{1}{(2\pi i)^2} \int_{t \in \Gamma} \int_{s \in \Sigma} \frac{t^\alpha}{s^\alpha} e^{\tau/t + 1/(2t^2) - \tau/s - 1/(2s^2)} e^{xt - ys} \frac{dt ds}{s - t}$$

A method behind these neat results?

RIEMANN-HILBERT METHOD

Let n be even. We look for $Y \in \mathbb{C}^{3\times 3}$ analytic in $\mathbb{C}\setminus\mathbb{R}$ such that

- on $\mathbb{R}_+$,

$$Y_+(x) = Y_-(x)\begin{pmatrix}1 & w_1(x) & w_2(x)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

- $Y(z)$ has the following behavior at infinity: as $z\to\infty$,

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix}z^n & 0 & 0\\ 0 & z^{-n/2} & 0\\ 0 & 0 & z^{-n/2}\end{pmatrix}, \quad z\in\mathbb{C}\setminus\mathbb{R}_+.$$

- as $z\to 0$, $z\in\mathbb{C}\setminus\mathbb{R}_+$,

$$Y(z) = \mathcal{O}\begin{pmatrix}1 & h(z) & 1\\ 1 & h(z) & 1\\ 1 & h(z) & 1\end{pmatrix}, \quad h(z) = \begin{cases}|z|^\alpha, & \text{if} \quad -1<\alpha<0,\\ \log|z|, & \text{if} \quad \alpha = 0,\\ 1, & \text{if} \quad 0<\alpha.\end{cases}$$

RIEMANN-HILBERT METHOD

[Bleher-Kuijlaars, 2004] The correlation kernel

$$K_n(x,y) = K_n(x,y;t)$$

can be computed as

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where Y is a solution of the mentioned RH problem.

THE R-H STEEPEST DESCENT ANALYSIS

Idea of the asymptotic analysis:

$$
\begin{array}{rcl}
 & \text{Start with } \mathbf{Y} & \\
 & \downarrow & \\
 & \mathbf{T} & \\
 & \downarrow & \\
 & \mathbf{U} & \\
 & \downarrow & \\
\text{conclude that} & & \\
\mathbf{I} + \text{ small} = & \mathbf{S} & \text{such that } \begin{cases} \mathbf{S}(\infty) = \mathbf{I} \\ \mathbf{S}_+(z) = \mathbf{S}_-(z)\,(\mathbf{I} + \text{ small}) \end{cases}
\end{array}
$$

THE R-H STEEPEST DESCENT ANALYSIS

Idea of the asymptotic analysis:

$$
\begin{array}{l}
\text{Start with } \mathbf{Y} \ \Rightarrow \text{ get an asymptotic expression for } \mathbf{Y} \\
\qquad\qquad\quad \uparrow \\
\qquad\qquad\quad \mathbf{T} \\
\qquad\qquad\quad \uparrow \\
\qquad\qquad\quad \mathbf{U} \\
\qquad\qquad\quad \uparrow
\end{array}
$$

$$
\begin{array}{r}
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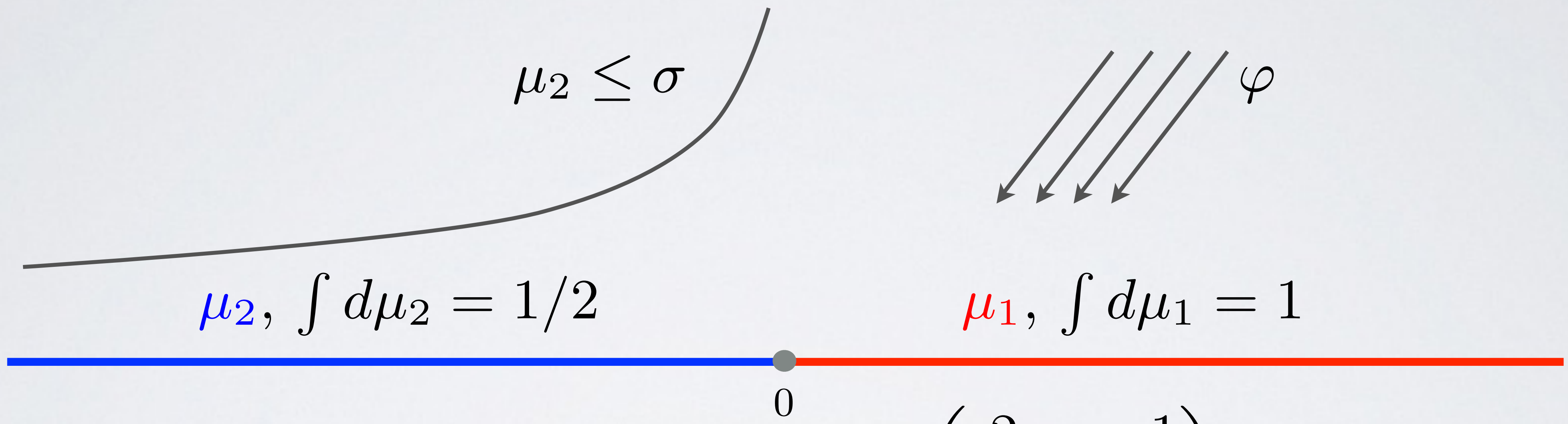
RIEMANN-HILBERT METHOD

A crucial step: normalization at infinity

Important ingredient: solving the vector logarithmic equilibrium.



Matrix of interaction $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$\frac{d\sigma}{dx} = \frac{\sqrt{a}}{\pi t}|x|^{-1/2}, \quad x \in (-\infty, 0]. \qquad \varphi(x) = \frac{x}{t(1-t)} - \frac{2\sqrt{ax}}{t}$$

Normalization at ∞ is done using the solution of this problem.

RIEMANN-HILBERT METHOD

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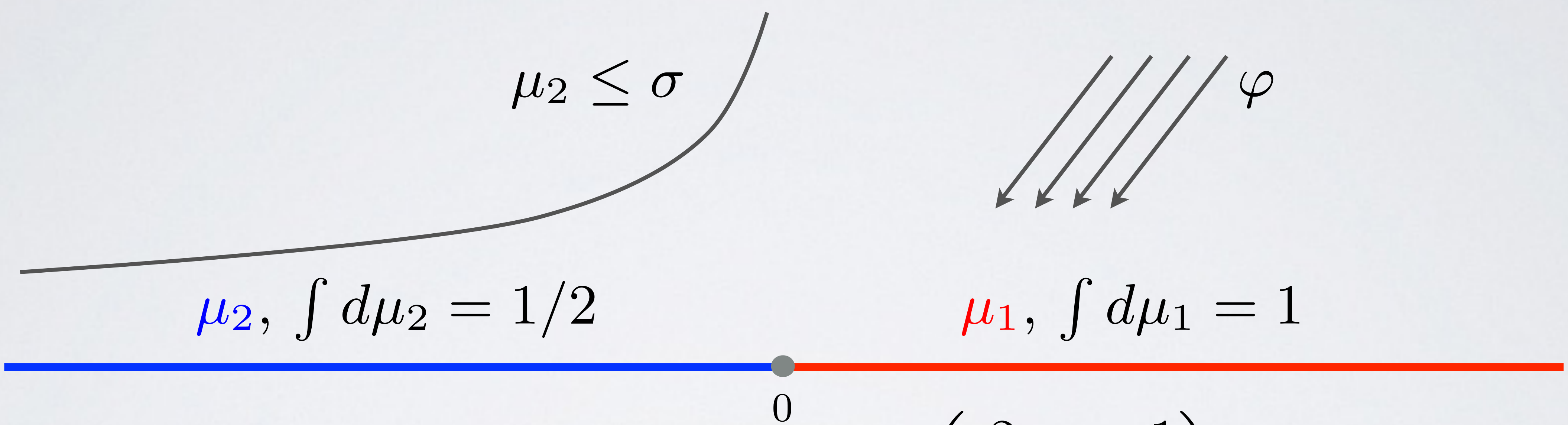

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This path leads us to beautiful landscapes of Riemann surfaces crossed by trajectories of quadratic differentials…

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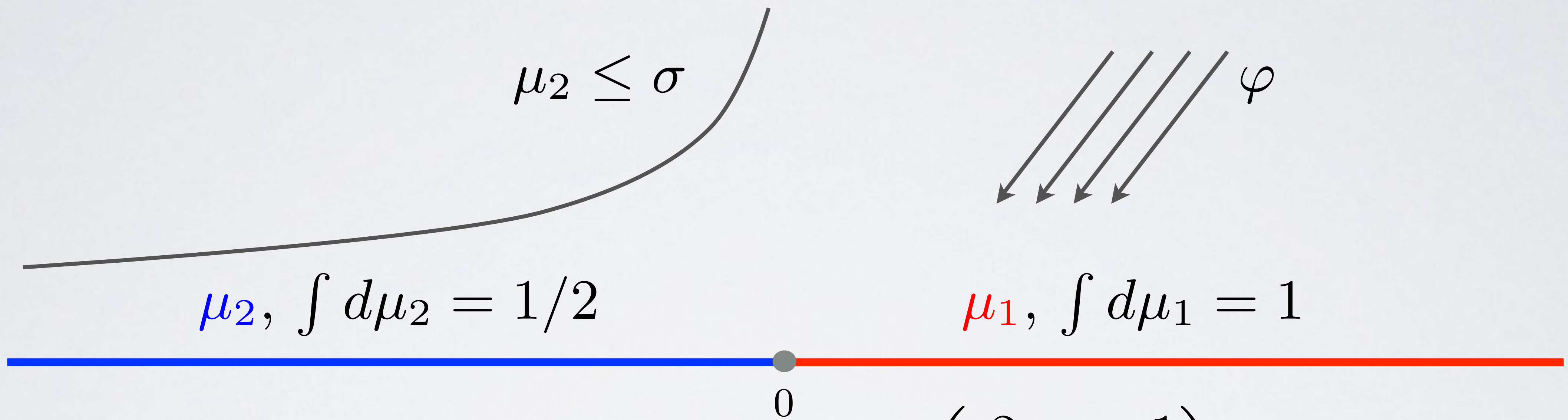

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But this is another story…


TO BE CONTINUED...