Deformation Theory and Moduli Spaces

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Brief history

- Kodaira-Spencer [1958] : modern approach in analytic category.
- Grothendieck's idea: Scheme theory is ideally suited for deformations and moduli in algebraic category.
- FGA [1960]: Descent. Formal schemes and existence theorem. Projective methods. Hilbert schemes. Picard schemes.
- Mumford's GIT [1965]. Projective quotients. Moduli for stable objects.
- Schlessinger [1965]: Functors on Artin Rings.
- Illusie [1968] Cotangent complex and application to lifting problems.
- Mumford [1964] : Moduli stack for elliptic curves. Deligne-Mumford stacks [1969].
- Artin [1968] : Approximation theorem. Étale gluing. Algebraic spaces.
- Artin [1974] : Introduction of algebraic stacks. Use of deformation theory to make moduli as algebraic stacks.

From deformations to Moduli: Overview - (i)

- What is a deformation? 'to prolong' or 'to extend' (speaking geometrically), or 'to lift' (speaking algebraically). This is the opposite of taking limits or specializations or reductions.
- We begin with a kind of **objects**. e.g. Varieties, line bundles on a variety, hypersurfaces in a given space, etc. Let E be such an object.
- **Parameter space** : a pointed space (T, t_0) where T is a scheme and t_0 is a (locally) closed point of T.
- A **deformation** of E parameterized by (T, t_0) is a **family** E_T of similar objects, together with an isomorphism $E \xrightarrow{\sim} E_{t_0}$.
- **Note**: A family E_T is more than just an indexed collection $(E_t)_{t \in T}$. The objects are held together in a 'continuous' manner. Flatness is the algebraic notion of importance here.

Overview - (ii). Infinitesimal theory.

- **Lifting to a square-zero thickening** is the fundamental step in generating infinitesimal lifts. Iteration gives higher order lifts.
- **Infinitesimal deformation** : parameterized by $T = \mathrm{Spec}\, A$, where A is an Artin local ring and $t_0 \in \mathrm{Spec}\, A$ is its closed point.
- A **tangent-obstruction** theory is about lifting a family from A to A' where $A' \to A$ is a quotient ring with nilpotent kernel. $|\mathrm{Spec}\, A| \subset |\mathrm{Spec}\, A'|$ is actually an equality.
- **Cotangent complexes** give tangent-obstruction theories.
- **Schlessinger theorem** Under suitable hypothesis, a limit over larger and larger infinitesimal deformations can give a **versal pro-deformation** (E_n) parameterized by $\mathrm{Spec}\, R$ for a complete local ring $R = \lim R/\mathfrak{m}^{n+1}$.

Overview - (iii). Algebraization and moduli stacks.

- A pro-deformation (E_n) over a complete local ring $R = \lim R/\mathfrak{m}^{n+1}$ in good cases gives a deformation $\mathcal{E}$ over R by the **Grothendieck existence theorem.**
- The deformation over R can by **Artin's approx. theorem** be approximated by a deformation over an **algebraic** ring R'. By **openness of versality** we get a nbd $U_E \subset \mathrm{Spec}\, R'$ on which the deformation of E is versal.
- By starting with all possible E, the resulting parameter spaces U_E of versal algebraic deformations can be glued together in **étale topology** to get an **algebraic space** as moduli. Works best when automorphisms are trivial.
- **Algebraic stacks** are a generalization of spaces which encodes automorphisms.
- **Artin's theorem [1974]**: **Moduli is an algebraic stack** under suitable (necessary and sufficient) hypothesis.

Moduli problems as functors -(i)

This is the view of moduli problems originating in Grothendieck [FGA].

- Let S be a noetherian, quasi-separated base scheme (for example, $S = \operatorname{Spec}\mathbb{C}$ or $\operatorname{Spec}\mathbb{Z}$).
- Category Aff/S of affine schemes over S :
  Objects $U \to S$ where U is affine.
  Morphisms are S-morphisms of schemes.
- Opposite category: $Rings/S$.
  Objects: $A/S = (A, \operatorname{Spec} A \to S)$.
  Morphisms: ring homomorphisms over S.
- Specially noteworthy objects of Aff/S are the **points over** S: these are morphisms $s : \operatorname{Spec} k \to S$ where k is a field.
- A moduli problem over S is given by a functor
  $\Phi : (Aff/S)^{op} \to Sets$, or equivalently, a functor
  $\Phi : Rings/S \to Sets$.

Moduli problems as functors -(ii)

- The **objects** of interest to the moduli problem are elements $E \in \Phi(k)$ for various $s : \mathrm{Spec}\, k \to S$ where k is a field. We say that such an E is defined over $s : \mathrm{Spec}\, k \to S$.
- For T/S, an element $F \in \Phi(T)$ is called as a **family parameterized by** T.
- The best desired solution to the moduli problem is a pair (M, P) consisting of a scheme (or an algebraic space) M/S together with a natural isomorphism $P : h_M \to \Phi$, where $h_M = Hom_S(-, M) : Aff/S \to Sets$ is the functor 'represented' by M. (Note that M need not be affine.)
- Such an M is called the moduli space. Usually (M, P) is written simply as M.
- **'Yoneda'**: $M \to S$ can be recovered uniquely up to a unique isomorphism from the functor h_M. (This is stronger than the usual Yoneda lemma of category theory as M need not be affine).

Moduli problems as functors -(iii)

- Grothendieck proved that for any scheme M/S, the functor $h_M : (Aff/S)^{opp} \to Sets$ satisfies **fpqc descent.**
- An **fpqc cover** (or an **étale cover** or a **Zariski cover**) of an object U in Aff/S is a finite collection of morphisms $(U_i \to U)_{i \in I}$ in Aff/S such that each $U_i \to U$ is flat (or étale or an open immersion) and U is the union of their images.
- A functor $\Phi : (Aff/S)^{op} \to Sets$ satisfies fpqc descent (or étale descent or Zariski descent) if for each fpqc cover (or étale cover or Zariski cover) $(U_i \to U)_{i \in I}$ in Aff/U, the following diagram is exact.

$$\Phi(U) \to \prod_i \Phi(U_i) \rightrightarrows \prod_{j,k} \Phi(U_j \times_U U_k)$$

  where the two maps on the right are induced by the two projections.
- Such a Φ is a **sheaf of sets** in fpqc or étale or Zariski topology.

Moduli problems as functors -(iv)

- If $\Phi : (\mathit{Aff}/S)^{opp} \to \mathit{Sets}$ is a sheaf in fpqc topology, then there is a **uniquely unique** extension of Φ to a functor
$\Phi' : (\mathit{Sch}/S)^{opp} \to \mathit{Sets}$ which is again a fpqc sheaf (suitably defined), where Sch/S is the category of all schemes over S, with Aff/S as a full subcategory.
Notation: we will denote Φ' simply by Φ.
- If $\Phi : (\mathit{Aff}/S)^{opp} \to \mathit{Sets}$ is not already a sheaf in étale topology or fppf topology, then we replace it by its étale or fppf sheafification $\Phi^{sh} : (\mathit{Aff}/S)^{opp} \to \mathit{Sets}$, which in good examples is also an fpqc sheaf.
(Comment on difficulty in fpqc sheafification, and its solution via universes.)
- For existence of a moduli, it is necessary but **not sufficient** that Φ is an fpqc sheaf. Example: $S = \mathrm{Spec}\,\mathbb{Z}$,
$\Phi : \mathit{Rings} \to \mathit{Sets} : R \mapsto R/2R$.

Parameter spaces as moduli: Examples -(i)

- In many moduli problems of interest, $\Phi(U)$ is the set of isomorphism classes of geometric objects parameterized by U. These geometric objects often have nontrivial automorphisms.
- However in some examples, **the only automorphisms** of the geometric objects **are trivial.** In such cases, the moduli space is commonly called as the 'parameter space'.
- Functor $\Phi : Rings \to Sets : R \mapsto SL_2(R)$.
- A parameter space exists: $SL_{2,\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}[a,b,c,d]/(ad-bc-1)$.
- $\Phi : Rings \to Sets$ defined by $\Phi(R) =$ the set of all rank 1 projective direct summand submodules $L \subset R^n$ (for a fixed n).
- The parameter space is $\mathbb{P}^n_{\mathbb{Z}}$ (projective n-space over $\mathrm{Spec}\,\mathbb{Z}$).
- $\Phi(R) =$ the set of all closed subschemes $X \subset \mathbb{P}^n_R$ such that X is flat over R.
- The parameter space is called the **Hilbert scheme.** Existence proved by Grothendieck in [FGA].

Parameter spaces as moduli: Examples -(ii)

- $\Phi(R) =$ the set of all equivalence classes of coherent quotients $q : \mathcal{O}^m_{\mathbb{P}^n_R} \to \mathcal{F}$ such that $\mathcal{F}$ is flat over R, where m and n are fixed integers. (Two quotients $q_1 : \mathcal{O}^m_{\mathbb{P}^n_R} \to \mathcal{F}_1$ and $q_2 : \mathcal{O}^m_{\mathbb{P}^n_R} \to \mathcal{F}_2$ are *equivalent* if there exists isom $\phi : \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_2$ such that $q_2 = \phi \circ q_1$.)
- The parameter space is called the **Quot scheme**. Existence proved by Grothendieck in [FGA].
- The Hilbert scheme is the special case of the Quot scheme when $m = 1$. The existence proof of Hilbert and Quot schemes is heavily dependent on techniques of projective geometry.
- Let $V \to S$ be a proper morphism, and let $\mathcal{E}$ be a coherent $\mathcal{O}_V$-module. For any R/S, let $\Phi(R)$ be the set of all equivalence classes of coherent quotients $q : \mathcal{E} \otimes_S R \to \mathcal{F}$ on X_R. This functor clearly satisfies fpqc descent.
- A parameter space (called Quot space) exists according to Artin (1968), however, we may to go **beyond** the cadre of schemes: the Quot space (and the 'Hilbert space') is an **algebraic space**.

What if there are nontrivial automorphisms -(i)

- If a geometric object has a non-trivial automorphism, it may be possible to have a family of such objects which locally trivial but globally nontrivial.
- For example, a vector space of dimension $n \geq 1$ has non-trivial automorphisms. Consequently, locally trivial but globally non-trivial vector bundles exist.
- Let $\Phi : Aff/S \to Sets$ associate to any U the set of all isomorphism classes of rank n vector bundles on U. Then Φ is not a sheaf in Zariski topology. For, if E is a non-trivial vector bundle on U, then on some open cover $(U_i)_{i \in I}$ it is trivial, so the map

$$\Phi(U) \to \prod_{i \in I} \Phi(U_i)$$

  is not injective.
- Sheafification of Φ produces the constant singleton sheaf – which is represented by S. But all information about the vector bundles (including the rank) is lost by the moduli S.

What if there are nontrivial automorphisms -(ii)

- However, there are many moduli problems which can be suitably restricted or rigidified so that all is not lost on sheafification.
- **Example** The moduli of line bundles on fibers of X/S (a proper flat morphism): when X/S is flat and projective with geometrically integral fibers, Grothendieck constructed the relative Picard scheme $Pic_{X/S}$ in [FGA].
  When X/S is flat, proper, and cohomologically flat in dimension 0, this was done by Artin (1968) by letting $Pic_{X/S}$ be an algebraic space.
- **Example** (Mumford): $X = V(x^2 + y^2 + tz^2) \subset \mathbb{P}^2_{\mathbb{R}[[t]]}$ over $S = \operatorname{Spec} \mathbb{R}[[t]]$.
- **Example** The moduli of **stable** vector bundles on a curve – Mumford 1962 (early success of GIT methods).
- **Example** The moduli of **pointed stable** curves (another success of GIT methods).
- In all these examples, note that the automorphisms are 'uniform'.

Moduli problems as S-groupoids -(i)

- Mumford found a truly dramatic way out by re-imagining how a moduli problem is to be posed, and what is its solution, when the objects to be classified admit non-trivial automorphisms: [Mumford 1963] *Picard groups of moduli problems*.
- A moduli problem over S is given by a category $\mathfrak{X}$ and a functor $\mathfrak{X} \to \mathit{Aff}/S$, which makes $\mathfrak{X}$ a groupoid over Aff/S (called as a 'groupoid over S' or an 'S-groupoid' for simplicity).
- A **groupoid** $(\mathfrak{X}, a)$ **over** S is by definition a category $\mathfrak{X}$ together with a functor $a : \mathfrak{X} \to \mathit{Aff}/S$ which satisfies (1) and (2) below.
- (1) For each S-morphism $\phi : U \to V$ and object F in $\mathfrak{X}$, there exists an object E in $\mathfrak{X}$ and a morphism $f : E \to F$ in $\mathfrak{X}$ such that $a(f) = \phi$.
- (2) Given $U \stackrel{\phi}{\to} V \stackrel{\psi}{\to} W$ in Aff/S, objects $E,\ F,\ G$ in $\mathfrak{X}$ respectively over $U,\ V,\ W$, and arrows $h : E \to G$ over $\psi \circ \phi$ and $g : F \to G$ over ψ, there exists a unique arrow $f : E \to F$ in $\mathfrak{X}$ over ϕ such that $g \circ f = h$.

Moduli problems as S-groupoids -(ii)

Let $\mathfrak{X}$ be an S groupoid (the notation for the functor $a : \mathfrak{X} \to Aff/S$ is usually left out for simplicity).

- Categorical fiber $\mathfrak{X}_U$ over U in Aff/S:
  Objects in $\mathfrak{X}_U$: all objects of $\mathfrak{X}$ which map to U.
  Morphisms in $\mathfrak{X}_U$:
  all morphisms in $\mathfrak{X}$ which map to id_U.
  If $U = \mathrm{Spec}\, A$ for an S-ring A/S, we may denote $\mathfrak{X}_U$ by $\mathfrak{X}_A$.
- It follows that each $\mathfrak{X}_U$ is a groupoid in the sense of being a category in which all morphisms are isomorphisms.
- For each S-morphism $\phi : U \to V$, we can choose a pullback functor $\phi^* : \mathfrak{X}_V \to \mathfrak{X}_U$, and a system of natural isomorphisms $\psi^*\phi^* \stackrel{\sim}{\to} (\phi\psi)^*$ which makes $U \mapsto \mathfrak{X}_U$ a pseudo-functor from Aff/S to the category of all groupoids. For notational simplicity, we will usually pretend that $\psi^*\phi^* = (\phi\psi)^*$.
- The data consisting of pullbacks ϕ^* and isomorphisms $\psi^*\phi^* \stackrel{\sim}{\to} (\phi\psi)^*$ is called a **cleavage** for a groupoid.

Big etale sheaf, or etale descent

- An **etale open cover** of an object U in Aff/S is a finite collection of morphisms $(U_i \to U)_{i \in I}$ in Aff/S such that each $U_i \to U$ is étale and U is the union of their images.
- A functor $\mathcal{F} : (Aff/S)^{op} \to Sets$ is called a **big étale sheaf** on S if for each étale open cover $(U_i \to U)_{i \in I}$ in Aff/U, the following diagram is exact.

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{j,k} \mathcal{F}(U_j \times_U U_k)$$

  where the two maps on the right are induced by the two projections.
- An S-groupoid $\mathfrak{X}$ is a **pre-stack** if given any U in Aff/S and E, F in $\mathfrak{X}_U$, the functor $Aff/U \to Sets : V \mapsto Hom_{\mathfrak{X}_V}(E, F)$ is a big étale sheaf on Aff/U (satisfies étale descent).

Effective etale descent

- An S-groupoid is said to satisfy **effective étale descent** if for each étale open cover $(U_i \to U)_{i \in I}$ in Aff/U, we have the following:
- Given any indexed collection of objects E_i in $\mathfrak{X}_{U_i}$ and isomorphisms $g_{ij} : E_j|_{U_{ij}} \to E_i|_{U_{ij}}$ with $g_{ij}g_{jk} = g_{ik}$ on U_{ijk}, there exists E in $\mathcal{F}(U)$ and isomorphisms $f_i : E|_{U_i} \to E_i$ such that $g_{ij} = f_i \circ f_j^{-1}$ on U_{ij} (notation: $U_{ij} = U_i \times_U U_j$ and the restrictions are the pullbacks under the two projections) which satisfy the cocyle condition $g_{ij}g_{jk} = g_{ik}$ on U_{ijk}, there exists E in $\mathcal{F}(U)$ and isomorphisms $f_i : E|_{U_i} \to E_i$ such that $g_{ij} = f_i \circ f_j^{-1}$ on U_{ij}.
- An S**-stack** is an S-prestack which satisfies effective étale descent.
- Any S-groupoid $\mathfrak{X}$ admits a functorial stackification, which is left adjoint to the inclusion of the category of all S-stacks into the category of all S-groupoids.

Algebraic stacks over S

- Recall that an S-stack $\mathfrak{X}$ is said to be **algebraic** if (i) the diagonal $\mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable, separated and quasicompact, and (ii) there exists an algebraic space X over S and a S-morphism $P : X \to \mathfrak{X}$ which is surjective and smooth.
- A moduli problem is *solved* if the S-groupoid $\mathfrak{X}$ is an **algebraic stack** over S.
- If our original moduli problem $\mathfrak{X}$, which is only an S-groupoid, is not even a stack (leave alone being an algebraic stack), then as the first step we replace $\mathfrak{X}$ by its stackification.
- The above is an important step in practice.
- After that we face the question whether $\mathfrak{X}$ is algebraic. Artin [1974] gives a theoretical answer to this via deformation theory.
- Explaining the above – to the extent possible within our time constraints – is the thrust of these lectures.

Moduli problem : Example BG

- Let G be a finite type separated smooth (or finite type separated flat) group-scheme over S. Example: $GL_{n,S}$ (or $\mu_{n,S}$).
- The S-groupoid $\mathfrak{X}$ has for objects all principal G-bundles E/U over all $U \in Aff/S$. A morphism $(f, \phi) : E/U \to F/V$ consists of $f : U \to V$ in Aff/S and an isomorphism $\phi : E \to f^*F$ of principal G-bundles over U.
- If $s : \mathrm{Spec}(k) \to S$ is a geometric point of S (that is, $k = \overline{k}$ is an algebraic closed field), then $\mathfrak{X}_s$ has only one object G_k up to isomorphism. It has $G(k)$ as its automorphism group in $\mathfrak{X}_s$.
- Given the presence automorphisms, the moduli cannot be a scheme (or an algebraic space).
- The moduli is an algebraic stack over S, denoted by BG. (Easy if G is smooth, needs some standard arguments in flat case).

Example: moduli for line bundles

- $\pi : X \to S$ a proper flat scheme over a noetherian base scheme S. Example X a complete complex variety, $S = \mathrm{Spec}\, C$.
- Moduli problem: An **object** – a line bundle L on X_s where $s : \mathrm{Spec}\, k \to S$ is a field valued point, and $X_s = \mathrm{Spec}\, k \times_S X$ the valued fiber.
  $\mathcal{L}$ on $X_T = X \times_S T$.
  Pullback $f^*\mathcal{L}$ on $X_{T'}$ under $f : T' \to T$.
- $Pic_{X/S} \to S$: the moduli space for line bundles on fibers of X/S (relative Picard 'scheme').
- **Questions** How to deform a given line bundle L on some X_s? Does $Pic_{X/S}$ exist? What are its local properties?

Example: Coherent sheaves

- $\pi : X \to S$ a proper family of schemes.
- Moduli problem: **objects** – coherent sheaves E of $\mathcal{O}_{X_s}$-modules on valued fibers X_s of X/S for $s : \mathrm{Spec}\, k \to S$.
- When $E \neq 0$, $\mathcal{O}(X_s)^\times \subset Aut(E)$.
  simple (this is the easier case, e.g. line bundles).
- A **family** parameterized by an S-scheme T : a coherent sheaf E_T on $X_T = X \times_S T$, which is **flat** over T. Pullback under $T' \to T$.
- **Questions** How to deform a coherent sheaf? Does a moduli space exist? What are its properties?
- **Variations** Vector bundles, Higgs bundles, connections, logarithmic connections, Λ-modules.

Example: First and second order infinitesimals

- **Dual numbers** Formal combinations $a + \epsilon b$, with $\epsilon^2 = 0$.
  Rings $\mathbb{R}[\epsilon]/(\epsilon^2)$, $\mathbb{C}[\epsilon]/(\epsilon^2)$, or $k[\epsilon]/(\epsilon^2)$ for any base field k.
- **Tangent vectors via dual numbers** X a variety, (x_i) local coordinates. Point $p \in X$: defined by $x_i \mapsto a_i$.
  Tangent vector $v = b_i \partial/\partial x_i \in T_p X$: defined by $x_i \mapsto a_i + \epsilon b_i$.
- **Example** Tangent vector to unit sphere $X = (\sum x_i^2 = 1)$ given by $\sum (a_i + \epsilon b_i)^2 = 1$. $\sum a_i^2 = 1$, $\sum a_i b_i = 0$ (assume $char(k) \neq 2$).
- **Example** Orthogonal group $O(n)$: ${}^t XX = I$.
  Tangent vector at $I \in O(n)$: ${}^t(I + \epsilon B)(I + \epsilon B) = I$. ${}^t B + B = 0$.
- $(I + \epsilon B)^{-1} = I - \epsilon B$.
- Lie algebra structure. $\mathbb{R}[s,t]/(s^2,t^2)$
  Commutator of $X = I + sB$ and $Y = I + tC$:
  $XYX^{-1}Y^{-1} = I + st(BC - CB)$.
  As $(st)^2 = 0$, $I + st(BC - CB)$ is tangent at I.
  This recovers the definition $[B, C] = BC - CB$.

Lifting vector bundles to a square-zero thickening -(i)

Assumption: Schemes are noetherian and separated.

- $X \subset X'$ closed subscheme, $\mathcal{I} \subset \mathcal{O}_{X'}$ its ideal sheaf.
- If $\mathcal{I}^2 = 0$ then X' is called a **square-zero thickening** of X. Then $\mathcal{I}$ is naturally a coherent $\mathcal{O}_X$-module.
- **Lifting homomorphisms** Let L', K' be line bundles on X', let $L = L'|_X$ and $K = K'|_X$. Let $\phi : L \to K$ be an $\mathcal{O}_X$-linear morphism. We want to lift ϕ to $\phi' : L' \to K'$.
- **Groupoid perspective** If $\phi'|_X : L \to K$ is an isomorphism, then so is $\phi' : L' \to K'$.
- $0 \to I \otimes_{\mathcal{O}_X} K \to K' \to K \to 0$ is exact. Apply $Hom_{X'}(L', -)$ to get long exact $0 \to H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \underline{Hom}(L, K)) \to Hom_{X'}(L', K') \to Hom_X(L, K) \xrightarrow{\partial} H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \underline{Hom}(L, K))$
- A lift exists if and only if $\partial(\phi) \in H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \underline{Hom}(L, K))$ is zero. This is the **obstruction** to lifting ϕ from X to X'.
- The set of all lifts of ϕ is a **principal set** under the action of $H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \underline{Hom}(L, K))$.

Lifting vector bundles to a square-zero thickening -(ii)

- **Lifting objects** Let L be a line bundle on X. We want to lift it to X'. A lift is a pair $(L', u : L \stackrel{\sim}{\to} L'|_X)$ up to equivalence, where L' is line bundle on X'.
- Special assumption for simplicity: the local and global restriction maps $\mathcal{O}_{X'}^\times \to \mathcal{O}_X^\times$ and $\mathcal{O}_{X'}^\times(X') \to \mathcal{O}_X^\times(X)$ are surjective. Hence,
- $0 \to \mathcal{I} \to \mathcal{O}_{X'}^\times \to \mathcal{O}_X^\times \to 0$ is exact, where $a \mapsto 1 + a$ under $\mathcal{I} \to \mathcal{O}_{X'}^\times$ (this uses $\mathcal{I}^2 = 0$). Also,
- $0 \to H^1(X, \mathcal{I}) \to H^1(X', \mathcal{O}_{X'}^\times) \to H^1(X, \mathcal{O}_X^\times) \stackrel{\partial}{\to} H^2(X, \mathcal{I})$ is exact.
- The element $\partial[L] \in H^2(X, \mathcal{I})$ is the **obstruction** to lifting L. A lift exists if and only if $\partial[L] = 0$.
- All lifts form a **principal set under** $H^1(X, \mathcal{I})$**-action.**
- **Infinitesimal automorphisms** of a lift $(L', u : L \stackrel{\sim}{\to} L'|_X)$: isomorphisms $\phi : L' \to L'$ with $u^{-1} \circ (\phi|_X) \circ u = \mathrm{id}_L$.
- These form the group $H^0(X, \mathcal{I})$ (exercise).

Lifting vector bundles to a square-zero thickening -(iii)

- We get $\mathit{Inf}_L(\mathcal{I}) = H^0(X, \mathcal{I})$, $\mathit{Tan}_L(\mathcal{I}) = H^1(X, \mathcal{I})$, and $\mathit{Obs}_L(\mathcal{I}) = H^2(X, \mathcal{I})$ in terms of general notation introduced later. Note that in this example, these are independent of L.
- Remove special assumptions on X. Let E be a vector bundle on X. Lifting problem under $X \hookrightarrow X'$. Cech computation – actually, an argument using gerbes – gives (exercise):
- A lift is possible if and only if an obstruction class $\mathit{obs}_{E,X,X'} \in \mathit{Obs}_E(\mathcal{I}) = H^2(X, \mathcal{I} \otimes \underline{\mathit{End}}(E))$ is zero.
- All lifts form a principal set under $\mathit{Tan}_E(\mathcal{I}) = H^1(X, \mathcal{I} \otimes \underline{\mathit{End}}(E))$.
- The infinitesimal automorphisms of any lift form the group $\mathit{Inf}_E(\mathcal{I}) = H^0(X, \mathcal{I} \otimes \underline{\mathit{End}}(E))$.

Lifting vector bundles to a square-zero thickening -(iv)

- **Lifting homomorphisms** Let E', F' be line bundles on X', let $E = E'|_X$, $F = F'|_X$ and $\phi : E \to F$ be an $\mathcal{O}_X$-linear homomorphism. We want to lift ϕ to $\phi' : E' \to F'$.
- **Groupoid perspective** If $\phi'|_X : E \to F$ is an isomorphism, then so is $\phi' : E' \to F'$.
- $0 \to \mathcal{I} \otimes_{\mathcal{O}_X} F \to F' \to F \to 0$ is exact. Apply $Hom_{X'}(E, -)$ to conclude:
- A lift ϕ' exists if and only if the **obstruction** $\partial(\phi) \in H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \underline{Hom}(E, F))$ is zero.
- The set of all lifts of ϕ is a principal set under $H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \underline{Hom}(E, F))$.

Lifting vector bundles to a square-zero thickening -(v)

- V variety over k, and $X_n = V \otimes_k k[t]/(t^{n+1})$ for any $n \geq 0$.
  X_n has the same underlying topological space as V.
  Regular functions on X_n are $f_0 + tf_1 + \ldots t^n f_n$ with $t^{n+1} = 0$.
- $X_n \subset X_{n+1}$ is a square-zero thickening with ideal sheaf $\mathcal{I}_n = (t^n) \subset \mathcal{O}_{X_{n+1}}$.
- For $n = 0$, we get $V = X_0 \subset X_1 = V[\epsilon] = V \otimes_k k[\epsilon]/(\epsilon^2)$.
  $\mathcal{I}_0 = (\epsilon) \subset \mathcal{O}_{V[\epsilon]}$ is isomorphic to $\mathcal{O}_V$ as a $\mathcal{O}_V$-module.
- Any E on V has a canonical lift $E[\epsilon]$ to $V[\epsilon]$.
  Hence $obs_{E,V,V[\epsilon]} = 0 \in Obs_E(\epsilon) = H^2(V, \underline{End}(E))$.
  (The cohomology itself may be non-zero.)
- $E[\epsilon]$ provides a base point. Hence all lifts to $V[\epsilon]$ form a vector space $Tan_E(\epsilon) = H^1(V, \underline{End}(E))$.
- The infinitesimal automorphisms form the vector space $Inf_E(\epsilon) = H^0(V, \underline{End}(E)) = End(E)$.
  Note All these spaces are finite dimensional if V/k is proper.

Square-zero lifts: coherent sheaves-(i)

- A vector bundle on X is the same as a coherent flat $\mathcal{O}_X$-module.
- Flatness is essential for a workable notion of a 'family'. Algebraic analog of continuity.
- Grothendieck invented the method of working in a **relative** set-up. A scheme X is replaced by a relative scheme $X \to S$. A **family** of coherent sheaves on fibers of $X \to S$ is a coherent $\mathcal{O}_X$-module that is flat as an $\mathcal{O}_S$-module.
- Given data: A surjection of rings $A' \to A$ with kernel J which is square-zero, and a scheme $X' \to \mathrm{Spec}\, A'$. We put $X = X' \otimes_{A'} A$.
- **Lifting homomorphisms** Given a pair of coherent sheaves $\mathcal{E}'$, $\mathcal{F}'$ on $X' \otimes_{A'} A$ which are flat over A', and an $\mathcal{O}_X$-homomorphism $\phi : \mathcal{E} = \mathcal{E}'|_X \to \mathcal{F}'|_X = \mathcal{F}$, we want $\mathcal{O}_{X'}$-homomorphism $\phi' : \mathcal{E}' \to \mathcal{F}'$ such that $\phi'|_X = \phi$.
- (Exercise) A lift exists if and only if obstruction $\partial(\phi) \in Ext^1_X(\mathcal{E}, J \otimes_A \mathcal{F})$ is zero.
- All lifts form a principal $Hom_X(\mathcal{E}, J \otimes_A \mathcal{F})$-set.

Square-zero lifts: coherent sheaves -(ii)

- **Lifting a coherent sheaf** Given a coherent sheaf $\mathcal{E}$ on $X = X' \otimes_{A'} A$, we want a pair $(\mathcal{E}', u : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'|_X)$, such that $\mathcal{E}'$ is a coherent $\mathcal{O}_{X'}$-module that is flat over A', and u is $\mathcal{O}_X$-linear isomorphism.
- Note that $i_*\mathcal{E}$ (where $i : X \hookrightarrow X'$) need not be A'-flat.
- (Exercise) A lift $(\mathcal{E}', u : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'|_X)$ exists if and only if an obstruction element $obs_{\mathcal{E},i} \in Ext^2_X(\mathcal{E}, J \otimes_A \mathcal{E})$ is zero.
- All lifts form a principal $Ext^1_X(\mathcal{E}, J \otimes_A \mathcal{E})$-set.
- form $Hom_X(\mathcal{E}, J \otimes_A \mathcal{E})$.
- Notice the occurrence of $Ext^i_X(\mathcal{E}, J \otimes_A \mathcal{E})$ for $i = 0, 1, 2$.

Square-zero lifts of schemes and morphisms - (i)

- **Prolonging morphisms** Let $g : Y \to X$ a morphisms of schemes over base S. $i : Y \hookrightarrow Y'$ closed embedding of schemes over S, defined by an ideal sheaf $J \subset \mathcal{O}_{Y'}$ with $J^2 = 0$. Makes J a coherent $\mathcal{O}_Y$-module. Commutative square:

$$\begin{array}{ccc} Y & \stackrel{g}{\to} & X \\ \downarrow & & \downarrow \\ Y' & \to & S \end{array}$$

- We want a north-east diagonal morphism $g' : Y' \nearrow X$ making the resulting diagram (square plus diagonal) commutative.
- If $X \to S$ is smooth, then a g' exists if and only if an obstruction $obs_{g,i} \in H^1(Y, J \otimes g^* T_{X/S})$ is zero. Here $T_{X/S}$ is the vertical tangent bundle, which is locally free as X/S is smooth.
- All prolongations form a principal $H^0(Y, J \otimes g^* T_{X/S})$-set.
- Automorphisms of any prolongation form the trivial group $0 = H^{-1}(Y, J \otimes g^* T_{X/S})$.

Square-zero lifts of schemes and morphisms - (ii)

- If X/S is not assumed smooth, we need a gadget called **cotangent complex** $L_{X/S}$ in the derived category $D^{\leq 0}(X)$.
- The answers $H^i(Y, J \otimes g^* T_{X/S})$ for $i = -1, 0, 1$ more generally become $Ext^i_Y(g^* L_{X/S}, J)$ respectively.
- **The category** $\mathfrak{Exal}(X/S, M)$: Let X/S be a scheme, M a quasicoherent $\mathcal{O}_X$-module.
- **Objects**: S-extensions $(i : X \hookrightarrow X', u)$ of X by M, where i is a closed embedding of S-schemes with square-zero ideal sheaf $\mathcal{I}$, together with a given $\mathcal{O}_X$-module isomorphism $u : M \xrightarrow{\sim} \mathcal{I}$. This gives an exact sequence of $\mathcal{O}_{X'}$-modules

$$0 \to M \xrightarrow{u} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where the homomorphism $u : M \to \mathcal{O}_{X'}$ is induced by $u : M \xrightarrow{\sim} \mathcal{I}$.

Square-zero lifts of schemes and morphisms - (iii)

- **Morphisms in** $\mathfrak{Exal}(X/S, M)$: A morphism ψ from (i_1, u_1) and (i_2, u_2) is a morphism $\psi : X'_1 \to X'_2$ which restricts to identity on X and gives a commutative diagram of abelian sheaves:

$$\begin{array}{ccccccc} 0 \to & M & \stackrel{u_1}{\to} \mathcal{O}_{X'_2} \to & \mathcal{O}_X & \to 0 \\ & \| & \psi^{\#} \downarrow & \| & \\ 0 \to & M & \stackrel{u_2}{\to} \mathcal{O}_{X'_1} \to & \mathcal{O}_X & \to 0 \end{array}$$

- By 5-lemma, ψ is an isomorphism, so $\mathfrak{Exal}(X/S, M)$ is a groupoid.
- $\mathfrak{Exal}(X/S, M)$ has a functorial addition, associativity, commutativity, unit and inverse, making it a Picard groupoid. (Name comes from the groupoid $\mathfrak{Pic}(X)$ of all line bundles on X.)
- Unit object: $X[M] = (|X|, \mathcal{O}_X \oplus M)$ with where $M^2 = 0$.
- Exercise: Give the functorial addition, associativity, etc.
- Isomorphism classes in $\mathfrak{Exal}(X/S, M)$ form a group $Exal(X/S, M)$.

Square-zero lifts of schemes and morphisms - (iv)

- Given any morphism of schemes $X \to S$, we have $L_{X/S}$ in $D^{\leq 0}(X)$. If $X \to S$ is finite type, then $L_{X/S}$ is a pseudo-coherent complex of flat $\mathcal{O}_X$-modules.
- $\mathcal{H}^0(L_{X/S}) = \Omega^1_{X/S}$. If X/S is smooth, then $L_{X/S}$ is just the sheaf $\Omega^1_{X/S}$ concentrated in degree 0.
- If $X \hookrightarrow X'$ is a closed embedding defined by ideal sheaf $\mathcal{I}$, then $\mathcal{H}^0(L_{X/X'}) = 0$ and $\mathcal{H}^{-1}(L_{X/X'}) = \mathcal{I}/\mathcal{I}^2$.
- $L_{Y/X}$ has perfect amplitude contained in $[-1, 0]$ if and only if $Y \to X$ is a l.c.i. morphism (smooth $\circ$ regular immersion).
- Morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ give a functorial exact triangle in $D^{\leq 0}(Z)$

$$g^* L_{Y/X} \to L_{Z/X} \to L_{Z/Y} \xrightarrow{(1)}$$

- This generalizes the exact sequence $g^*\Omega^1_{Y/X} \to \Omega^1_{Z/X} \to \Omega^1_{Z/Y} \to 0$ as well as the exact sequence $\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega^1_{Y/X}|_Z \to \Omega^1_{Z/X} \to 0$ when $Z \hookrightarrow Y$ is a closed embedding of X-schemes.

Square-zero lifts of schemes and morphisms - (v)

- **Important fact** $Exal(X/S, M) = Ext^1(L_{X/S}, M)$.
- Automorphism group of $(i : X \hookrightarrow X', u : M \xrightarrow{\sim} \ker(i))$ in $\mathfrak{Exal}(X/S, M)$ is $Der(X/S, M) = Hom(\Omega^1_{X/S}, M) = Ext^0(L_{X/S}, M)$.
- **Lifting a scheme** Given $X \hookrightarrow X'$ a square-zero extension with ideal $\mathcal{I}$, $f : Y \to X$ a flat morphism, we want a square-zero extension $Y \hookrightarrow Y'$ and a flat morphism $f' : Y' \to X'$ such that the following diagram is cartesian.

$$\begin{array}{ccc} Y & \hookrightarrow & Y' \\ f \downarrow & & \downarrow f' \\ X & \hookrightarrow & X' \end{array}$$

- The X'-extension $(i : X \hookrightarrow X', \mathrm{id} : \mathcal{I} \to \ker(i))$ of X by $\mathcal{I}$ defines an element $[X \hookrightarrow X'] = [(i, \mathrm{id})] \in Exal(X/X', \mathcal{I}) = Ext^1(L_{X/X'}, \mathcal{I})$.
- Under the adjunction $a : \mathcal{I} \to f_* f^* \mathcal{I}$, this defines an element $a_*[X \hookrightarrow X'] \in Exal(X/X', f_* f^* \mathcal{I}) = Ext^1(L_{X/X'}, f_* f^* \mathcal{I}) = Ext^1(f^* L_{X/X'}, f^* \mathcal{I}) = Hom(f^* \mathcal{I}, f^* \mathcal{I})$. Exercise: $a_*[X \hookrightarrow X'] = \mathrm{id}_{f^* \mathcal{I}}$.

Square-zero lifts of schemes and morphisms - (vi)

- The equality $Ext^1(f^*L_{X/X'}, f^*\mathcal{I}) = Hom(f^*\mathcal{I}, f^*\mathcal{I})$ used flatness of f and the facts that $\mathcal{H}^0(L_{X/X'}) = 0$, $\mathcal{H}^{-1}(L_{X/X'}) = \mathcal{I}$.
- A lift $(j : Y \hookrightarrow Y, f' : Y' \to X')$ gives an isomorphism $v : f^*\mathcal{I} \to \ker(j)$ as f' is flat and as $f = f'|_X$ (cartesian condition).
- This defines an element $[(j, v)] \in Exal(Y/X', f^*\mathcal{I}) = Ext^1(L_{Y/X'}, f^*\mathcal{I})$ with the following property:
- Under the natural map $Exal(Y/X', f^*\mathcal{I}) \to Exal(X/X', f^*\mathcal{I}) = End(f^*\mathcal{I})$, we have $[(j, v)] \mapsto a_*[X \hookrightarrow X'] = \mathrm{id}_{f^*\mathcal{I}}$.
- Conversely, if $\alpha \in Exal(Y/X', f^*\mathcal{I})$ maps to $a_*[X \hookrightarrow X']$, then α defines a flat lift of $X \hookrightarrow X'$ as desired. For, if α is the class of $(j : Y \hookrightarrow Y', f' : Y' \to X')$, then f' is flat by the following:
- **Square-zero criterion for flatness** $\mathcal{I} \subset A'$ ideal, $\mathcal{I}^2 = 0$. M' an A'-module such that $M = M'/\mathcal{I}M'$ is a flat $A = A'/\mathcal{I}$-module and the natural map $\mathcal{I} \otimes_A M \to M$ is injective. Then M' is flat over A'.

Square-zero lifts of schemes and morphisms - (vii)

- Consider the exact triangle $f^* L_{X/X'} \to L_{Y/X'} \to L_{Y/X} \xrightarrow{(1)}$.
  Apply $Hom(-, f^*\mathcal{I})$ to get exact $0 \to Ext^1(L_{Y/X}, f^*\mathcal{I}) \to Ext^1(L_{Y/X'}, f^*\mathcal{I}) \to Ext^1(f^* L_{X/X'}, f^*\mathcal{I}) \xrightarrow{\partial} Ext^2(L_{Y/X}, f^*\mathcal{I})$.
  It begins with zero as $Ext^0(f^* L_{X/X'}, f^*\mathcal{I}) = 0$ as $\mathcal{H}^0(f^* L_{X/X'}) = f^*\Omega^1_{X/X'} = 0$ as f is flat.
- Making the substitutions $Ext^1(L_{Y/X'}, f^*\mathcal{I}) = Exal(Y/X', f^*\mathcal{I})$ and $Ext^1(f^* L_{X/X'}, f^*\mathcal{I}) = End(f^*\mathcal{I})$, we get an exact sequence
  $0 \to Ext^1(L_{Y/X}, f^*\mathcal{I}) \to Exal(Y/X', f^*\mathcal{I}) \to End(f^*\mathcal{I}) \xrightarrow{\partial} Ext^2(L_{Y/X}, f^*\mathcal{I})$.
- $X \hookrightarrow X'$ admits a lift if and only if $\mathrm{id}_{f^*\mathcal{I}}$ lies in the image of $Exal(Y/X', f^*\mathcal{I}) \to End(f^*\mathcal{I})$.
- Lifts are the same as pre-images of $\mathrm{id}_{f^*\mathcal{I}}$ in $Exal(Y/X', f^*\mathcal{I})$.

Square-zero lifts of schemes and morphisms - (viii)

- Therefore the element $\partial(\mathrm{id}_{f^*\mathcal{I}}) \in \mathit{Ext}^2(L_{Y/X}, f^*\mathcal{I})$ is zero if and only if a flat lift of $X \hookrightarrow X'$ exists. Obstruction.
- All lifts form a principal $\mathit{Ext}^1(L_{Y/X}, f^*\mathcal{I})$-set.
- The group of automorphisms of any lift is $\mathit{Ext}^0(L_{Y/X}, f^*\mathcal{I})$.
- The lifting problem for morphisms is the fundamental lifting problem for all those moduli problems which are representable by an algebraic stack. For, an family x over T is a 1-morphism $x : T \to \mathfrak{X}$, and we wish to prolong x to $x' : T' \to \mathfrak{X}$ where $T \hookrightarrow T'$ is an infinitesimal extension of schemes.
- However, it was noticed much earlier that many others lifting problems reduce to the lifting problem for schemes and morphisms by clever tricks – see [Illusie]. For example (Nagata), a $\mathbb{Z}/(2)$-graded version gives the lifting of coherent sheaves.

Square-zero lifts of schemes and morphisms - (ix)

- A stacky version of the cotangent complex exists due to Laumon (see [L-MB] chapter 17). A crucial difference is $L_{\mathfrak{X}/S}$ is in $D^{\leq 1}(\mathfrak{X})$. Exercise: Determine $L_{BG/k}$ for $G = GL_{n,k}$.
- Theorem [Olsson 2006]: given a 1-morphism $x : T \to \mathfrak{X}$ from a scheme T to an algebraic stack over a base S, and square zero extension of S-schemes $T \hookrightarrow T'$ defined by an ideal J, we have:
- The obstruction to the existence of a lift is an element $obs_x \in Ext^1(x^* L_{\mathfrak{X}/S}, J) = Obs_x(J)$.
- The set of all lifts is a principal set under $Ext^0(x^* L_{\mathfrak{X}/S}, J) = Tan_x(J)$.
- The infinitesimal automorphisms of a lift are $Ext^{-1}(x^* L_{\mathfrak{X}/S}, J) = Inf_x(J)$.

Illusie : *Complexe cotangent et déformations* I, II. Springer LNM 239 (1971), 283 (1972). *Cotangent complex and deformations of torsors and group schemes*. LNM 274 (1972).
Olsson : *Deformation theory of representable morphisms of algebraic stacks*. Math. Z. 253 (2006).

Artin local rings: the category $Art_{k/\Lambda}$

that systematically parameterizes **infinitesimal deformations**.

- An **Artin local ring** A is a noetherian ring which has a unique prime ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is maximal and $\mathfrak{m}^{n+1} = 0$ for some $n \geq 0$. Examples: Any field k, $k[x,y,z]/(x^2,y^3,z^5)$, $\mathbb{Z}/(4)$.
- A noetherian local ring $(R,\mathfrak{n})$ is **complete** if $R = \lim_{\leftarrow}(R/\mathfrak{n}^{n+1})$. Examples: p-adic integers $\mathbb{Z}_p$, formal power series $k[[s,t]]$. Not complete (but henselian): convergent power series $\mathbb{C}\{z\}$.
- General set-up: $(\Lambda, \mathfrak{m}_\Lambda)$ a complete noetherian local ring, $\Lambda/\mathfrak{m}_\Lambda = \kappa$ its residue field, k/κ a given finite extension field. $Art_{k/\Lambda}$ category of Artin local Λ-algebras A with residue field k. **Note:** The finite extension k/κ need not be separable.
- Objects: $(A, \Lambda \xrightarrow{\phi} A, A/\mathfrak{m}_A \xrightarrow{\psi} k)$, where A Artin local ring, ϕ ring homomorphism, $\phi(\mathfrak{m}_\Lambda) \subset \mathfrak{m}_A$, and ψ is an isomorphism over Λ. Notation: simply A.
  Arrows: Local ring homomorphisms preserving ϕ, ψ.
- Examples: $Art_{\mathbb{C}/\mathbb{C}}$, $Art_{\mathbb{C}/\mathbb{R}[[s,t]]}$, $Art_{\mathbb{F}_p(t^{1/p})/\mathbb{F}_p(t)}$, $Art_{\mathbb{F}_p/\mathbb{Z}_p}$ (FLT).

The categories $Art_{k/\Lambda}$ and $\widehat{Art}_{k/\Lambda}$

- For any finite k-vector space V, we get an object $k[V] = k \oplus V$ of $Art_{k/\Lambda}$. Makes $FinVect_k$ a full subcategory of $Art_{k/\Lambda}$.
- k is final object of $Art_{k/\Lambda}$. Fiber products exist in $Art_{k/\Lambda}$. $k[V] \times_k k[W] = k[V \oplus W]$.
- Category $\widehat{Art}_{k/\Lambda}$: objects R are complete local noetherian Λ-algebras ($\mathrm{im}(\mathfrak{m}_\Lambda) \subset \mathfrak{m}_R$) together with a given Λ-isomorphism $R/\mathfrak{m}_R \to k$.
- $Art_{k/\Lambda}$ is a full subcategory of $\widehat{Art}_{k/\Lambda}$. If R is in $\widehat{Art}_{k/\Lambda}$ then each $R_n = R/\mathfrak{m}_R^{n+1}$ is in $Art_{k/\Lambda}$ for $n \geq 0$, and $R = \lim_{\leftarrow} R_n$.
- For a groupoid $\mathfrak{X}$ over $Art_{k/\Lambda}^{opp}$, a **formal object** over R is a sequence $(E_n, \phi_n)_{n\geq 0}$ where E_n is in $\mathfrak{X}(R_n)$ and $\phi_n : E_n \xrightarrow{\sim} E_{n+1}|_{R_n}$. Morphisms between two formal objects over R: obvious definition.
- Exercise ('Yoneda'): The natural map $Hom(h_R, F) \to \widehat{F}(R)$ is a bijection for $F : Art_{k/\Lambda} \to Sets$ and R in $\widehat{Art}_{k/\Lambda}$.

The infinitesimal conditions of Artin -(i)

- We choose a base scheme S and a subcategory $\mathcal{C}$ of $Rings/S$. The category $\mathcal{C}$ is supposed to be closed under the operations that we will subsequently need. Example: $S = \Lambda$ and $\mathcal{C} = Art_{k/\Lambda}$.
- A **deformation situation** in $\mathcal{C}$ consists of the following data :
  A_0 a ring in $\mathcal{C}$,
  M an A_0-module,
  $A' \twoheadrightarrow A \twoheadrightarrow A_0$ surjections in $\mathcal{C}$ with nilpotent kernels, with $\ker(A' \twoheadrightarrow A_0) \ker(A' \twoheadrightarrow A) = 0$,
  $M \xrightarrow{\sim} \ker(A' \twoheadrightarrow A)$ an A_0-module isomorphism.
- By abuse of notation, we will refer to a 'deformation situation $(A' \twoheadrightarrow A \twoheadrightarrow A_0, M)$ in $\mathcal{C}$'.
- In [Schlessinger 1966], we have $\mathcal{C} = Art_{k/\Lambda}$. The most common deformation situation that is considered there has $A_0 = k$, $A' \twoheadrightarrow A'/I = A$ any quotient in $Art_{k/\Lambda}$ with $\mathfrak{m}_{A'} I = 0$, and M the resulting finite dimensional k-vector space I.

The infinitesimal conditions of Artin -(ii)

- Let $\mathfrak{X}$ be a groupoid over $\mathcal{C}^{opp}$. Let $A \in Ob\,\mathcal{C}$, and let $a \in Ob\,\mathfrak{X}(A)$. Let $\mathcal{C}_{/A}$ be the category of all $B \to A$ in $\mathcal{C}$ (the comma category). We define a groupoid $\mathfrak{X}_a$ on $\mathcal{C}_{/A}^{opp}$ as follows.
- For any $f : B \to A$ in $\mathcal{C}$, we define $\mathfrak{X}_a(f : B \to A)$ to be the category whose objects are arrows $a \to b$ in $\mathfrak{X}$ over the morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ (we will simply say 'over $f : B \to A$').
- When the homomorphism $f : B \to A$ is understood, we will denote $\mathfrak{X}_a(f : B \to A)$ simply by $\mathfrak{X}_a(B)$.
- In terms of a cleavage, the objects of $\mathfrak{X}_a(B)$ are pairs $(b, a \stackrel{\sim}{\to} b|_A)$, where $b \in Ob\,\mathfrak{X}(B)$, $b|_A \in Ob\,\mathfrak{X}(A)$ is the 'pullback' under $f : B \to A$ in terms of the chosen cleavage, and $a \stackrel{\sim}{\to} b|_A$ is an isomorphism in $\mathfrak{X}(A)$.
- Let $f : B \to A$ be a homomorphism in $\mathcal{C}$, and let $\phi : a \to b_1$ and $\psi : a \to b_2$ be objects of $\mathfrak{X}_a(B)$ over f. A morphism in $\mathfrak{X}_a(B)$ from $a \to b_1$ to $a \to b_2$ is a morphism $\eta : b_1 \to b_2$ in $\mathfrak{X}(B)$ (lying over id_B) such that $\eta \circ \phi = \psi$.

The infinitesimal conditions of Artin -(iii)

- In particular, for $\mathrm{id} : A \to A$, the groupoid $\mathfrak{X}_a(A)$ is **trivial**: it consists of a single object (a, id_a) whose only automorphism is identity.
- Let $\overline{\mathfrak{X}_a}(B)$ denote the set of isomorphism classes in $\mathfrak{X}_a(B)$. Then $B \mapsto \overline{\mathfrak{X}_a}(B)$ defines a **functor** $\overline{\mathfrak{X}_a} : \mathcal{C}_{/A} \to$ *Sets*.
- Let $\overline{\mathfrak{X}}(B)$ denote the set of isomorphism classes in $\mathfrak{X}(B)$. Then $B \mapsto \overline{\mathfrak{X}}(B)$ defines a functor $\overline{\mathfrak{X}} : \mathcal{C} \to$ *Sets*.
- **Caution!** Given $(a) \in \overline{\mathfrak{X}}(A)$, we can make a **another functor** $(\overline{\mathfrak{X}})_{\overline{a}} : \mathcal{C}_{/A} \to$ *Sets* starting from $\overline{\mathfrak{X}}$, by associating to $B \to A$ the subset $(\overline{\mathfrak{X}})_{\overline{a}}(B) \subset \overline{\mathfrak{X}}(B)$ which consists of $\overline{b} \in \overline{\mathfrak{X}}(B)$ such that $\overline{b}|_A = \overline{a}$. This is **not** the functor $\overline{\mathfrak{X}_a}$ in general.
- Thus, even when we want to study infinitesimal deformation theory for set-valued functors on $Art_{k/\Lambda}$ (as Schlessinger did), we must begin with a groupoid $\mathfrak{X}$ over $Art_{k/\Lambda}$.
- The functor $\overline{\mathfrak{X}_a}$ is not made from the functor $\overline{\mathfrak{X}} : \mathcal{C}_{/A} \to$ *Sets*. We need the groupoid $\mathfrak{X}$ to make both $\mathfrak{X}_a$ and $\overline{\mathfrak{X}_a}$.

The infinitesimal conditions of Artin -(iv)

The condition **(S-1 a, b)**.

- Begin with a deformation situation $(A' \twoheadrightarrow A \twoheadrightarrow A_0,\ M)$ in $\mathcal{C}$. Assume $\mathcal{C}$ has all fibered products. Let $\mathfrak{X}$ be a groupoid over $\mathcal{C}^{opp}$.
- **(S-1 a)** Let $B \to A$ be in $\mathcal{C}$, such that the composite $B \to A \to A_0$ is surjective. Let $a \in Ob\,\mathfrak{X}(A)$. Then the induced map of sets

$$\overline{\mathfrak{X}_a}(A' \times_A B) \to \overline{\mathfrak{X}_a}(A') \times \overline{\mathfrak{X}_a}(B)$$

  is surjective.
- *Notation*: R a ring, M an R-module. $R[M]$ denotes the R-algebra $R \oplus M$ with $r \mapsto (r, 0)$. $(r, m)(r', m') = (rr', rm' + r'm)$. $M^2 = 0$.
- **(S-1 b)** Let $B \twoheadrightarrow A_0$ be a surjection in $\mathcal{C}$, with A_0 reduced. Let $a_0 \in Ob\,\mathfrak{X}(A_0)$. Let M be a finite A_0-module. Note that $B[M] = B \times_{A_0} A_0[M]$. Then the induced map of sets

$$\overline{\mathfrak{X}_{a_0}}(B[M]) = \overline{\mathfrak{X}_{a_0}}(B \times_{A_0} A_0[M]) \to \overline{\mathfrak{X}_{a_0}}(B) \times \overline{\mathfrak{X}_{a_0}}(A_0[M])$$

  is bijective.

The infinitesimal conditions of Artin -(v)

- With notation as in (S-1 b) above, any object of $\mathfrak{X}_{a_0}(B)$ has the form $(b, u : a_0 \to b|_{A_0})$ where $b \in Ob\,\mathfrak{X}(B)$. Hence we have

$$\overline{\mathfrak{X}_{a_0}}(B[M]) = \coprod_{(b,u)\in\overline{\mathfrak{X}_{a_0}}(B)} \overline{\mathfrak{X}_b}(B[M])$$

  Hence (S 1 b) has the following alternative form.

- **(S-1 b)** Let $B \twoheadrightarrow A_0$ be a surjection in $\mathcal{C}$, with A_0 reduced. Let M be a finite A_0-module, and let $B[M] \to A_0[M]$ be the induced surjection in $\mathcal{C}$. Let $b \in Ob\,\mathfrak{X}(B)$, and let $a_0 = b|_{A_0} \in Ob\,\mathfrak{X}(A_0)$ be its restriction. Then the induced map of sets

$$\overline{\mathfrak{X}_b}(B[M]) \to \overline{\mathfrak{X}_{a_0}}(A_0[M])$$

  is a bijection.

The infinitesimal conditions of Artin -(vi)

The following condition, called **(S-1')** by Artin, is stronger than (S-1) but weaker than the Rim-Schlessinger condition (R-S).

- **(S-1')** With notation as in (S-1)(a), the induced functor

$$\mathfrak{X}(A' \times_A B) \to \mathfrak{X}(A') \times_{\mathfrak{X}(A)} \mathfrak{X}(B)$$

  is an equivalence of groupoids, where the right hand side is the fiber product of groupoids.
- Equivalently, for each $a \in Ob\, \mathfrak{X}(A)$, the induced functor

$$\mathfrak{X}_a(A' \times_A B) \to \mathfrak{X}_a(A') \times \mathfrak{X}_a(B)$$

  is an equivalence of groupoids, where the right hand side is the direct product of groupoids.
- Exercise: Show that (S-1') $\Rightarrow$ (S-1) (a) and (b).

The Rim-Schlessinger condition

- Let $\mathfrak{X}$ be an S-groupoid. The following is called the **Rim-Schlessinger** condition.
  (R-S) If $A' \to A$ is a surjection in $Rings/S$ with nilpotent kernel and $B \to A$ any homomorphism in $Rings/S$, then the natural functor

$$\mathfrak{X}(A' \times_A B) \to \mathfrak{X}(A') \times_{\mathfrak{X}(A)} \mathfrak{X}(B)$$

  is an equivalence of categories.
- The condition (S-1') of Artin is weaker than this, as it assumes that the induced map $B \to A/Nil(A)$ is surjective.
- Another weaker version of the Rim-Schlessinger condition is when in the above, A, A', B are supposed to be Artin local, such that the homomorphisms induce isomorphisms on residue fields.

The infinitesimal conditions of Artin -(vii)

- (S-1)(b) implies that the map ϕ below is bijective:

$$\overline{\mathfrak{X}_{a_0}}(A_0[M \oplus M]) = \overline{\mathfrak{X}_{a_0}}(A_0[M] \times_{A_0} A_0[M]) \stackrel{\phi}{\to} \overline{\mathfrak{X}_{a_0}}(A_0[M]) \times \overline{\mathfrak{X}_{a_0}}(A_0[M])$$

- This gives rise to a natural addition on $\overline{\mathfrak{X}_{a_0}}(A_0[M])$ as the composite

$$\overline{\mathfrak{X}_{a_0}}(A_0[M]) \times \overline{\mathfrak{X}_{a_0}}(A_0[M]) \stackrel{\phi^{-1}}{\to} \overline{\mathfrak{X}_{a_0}}(A_0[M \oplus M]) \to \overline{\mathfrak{X}_{a_0}}(A_0[M])$$

  where the last map is induced by $+ : M \oplus M \to M$.
- For any $\lambda \in A_0$, the scalar multiplication $\lambda : M \to M$ gives A_0-algebra homomorphism $A_0[M] \to A_0[M] : (a, m) \mapsto (a, \lambda m)$. This induces $\lambda : \overline{\mathfrak{X}_{a_0}}(A_0[M]) \to \overline{\mathfrak{X}_{a_0}}(A_0[M])$. This makes $\overline{\mathfrak{X}_{a_0}}(A_0[M])$ an A_0-module.
- Notation: $D_{a_0}(M) = \overline{\mathfrak{X}_{a_0}}(A_0[M])$ as an A_0-module. This is functorial in (a_0, M), and depends linearly on (A_0, M).

The infinitesimal conditions of Artin -(viii)

- (S-1) implies that in any deformation situation $(A' \twoheadrightarrow A \overset{q_0}{\twoheadrightarrow} A_0,\, M)$, for any object $a \in Ob\,\mathfrak{X}(A)$ and its restriction $a_0 \in Ob\,\mathfrak{X}(A_0)$, the group $\overline{\mathfrak{X}_{a_0}}(A_0[M])$ acts transitively on the set $\overline{\mathfrak{X}_a}(A')$, as follows.
- We have an isomorphism $A' \times_{A_0} A_0[M] \overset{\sim}{\to} A' \times_A A'$ defined by $(r', r, m) \mapsto (r', r' + m)$.
- Hence we have a bijection $\overline{\mathfrak{X}_{a_0}}(A') \times \overline{\mathfrak{X}_{a_0}}(A_0[M]) \overset{(S\,1\,b)}{=} \overline{\mathfrak{X}_{a_0}}(A' \times_{A_0} A_0[M]) = \overline{\mathfrak{X}_{a_0}}(A' \times_A A')$.
- Observe that
$$\overline{\mathfrak{X}_{a_0}}(A') = \coprod_{(a,u)\in\overline{\mathfrak{X}_{a_0}}(A)} \overline{\mathfrak{X}_a}(A')$$
and similarly,
$$\overline{\mathfrak{X}_{a_0}}(A' \times_A A') = \coprod_{(a,u)\in\overline{\mathfrak{X}_{a_0}}(A)} \overline{\mathfrak{X}_a}(A' \times_A A')$$

The infinitesimal conditions of Artin -(ix)

- By (S-1 a) we have a surjection $\overline{\mathfrak{X}_a}(A' \times_A A') \to \overline{\mathfrak{X}_a}(A') \times \overline{\mathfrak{X}_a}(A')$. Hence from the above disjoint unions we get the required surjection

$$\overline{\mathfrak{X}_a}(A') \times \overline{\mathfrak{X}_{a_0}}(A_0[M]) \to \overline{\mathfrak{X}_a}(A') \times \overline{\mathfrak{X}_a}(A')$$

  of the form (p_1, α) which defines a transitive action α.
- Exercise: verify that α is indeed an action. The action α is transitive as the map (p_1, α) is surjective. If the S-groupoid $\mathfrak{X}$ satisfies (R-S) ((S-1') is enough), then $\overline{\mathfrak{X}_a}(A' \times_A A') = \overline{\mathfrak{X}_a}(A') \times \overline{\mathfrak{X}_a}(A')$, so the above action is both transitive and free.
- Condition (S-1) is called as **semi-homogeneity** and (S-1') as **homogeneity** in Rim [SGA7].
- **Condition (S-2)**: $D_{a_0}(M) = \overline{\mathfrak{X}_{a_0}}(A_0[M])$ is a **finite A_0-module**.

More on (R-S) -(i)

The condition that $\mathfrak{X}(A' \times_A B) \to \mathfrak{X}(A') \times_{\mathfrak{X}(A)} \mathfrak{X}(B)$ is an equivalence of groupoids is made of two requirements:

- **Full faithfulness**: Let $c_1, c_2 \in Ob\,\mathfrak{X}(A' \times_A B)$. Then the natural map below is a bijection.

  $$Hom(c_1, c_2) \to Hom(c_1|_{A'}, c_2|_{A'}) \times_{Hom(c_1|_A, c_2|_A)} Hom(c_1|_B, c_2|_B)$$

  In particular, for any $c \in Ob\,\mathfrak{X}(A' \times_A B)$, we get an isomorphism

  $$Aut(c) \to Aut(c|_{A'}) \times_{Aut(c|_A)} Aut(c|_B)$$

- **Essential surjectivity**: The natural map below is surjective.

  $$\overline{\mathfrak{X}}(A' \times_A B) \to \overline{\mathfrak{X}}(A') \times_{\overline{\mathfrak{X}}(A)} \overline{\mathfrak{X}}(B)$$

More on (R-S) -(ii)

- Recall the condition of representability of the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$: given any $u : U \to \mathfrak{X}$ and $v : V \to \mathfrak{X}$ where U and V are in Aff/S, the S-groupoid fiber product $U \times_{\mathfrak{X}} V$ should be representable by an algebraic space over S.
- The (R-S) condition on $\mathfrak{X}$ is an input in the proof that the diagonal of $\mathfrak{X}$ is representable, by the following 'bootstrap' argument:
- **Exercise**: Show that the 'fully faithful' part of the condition (R-S) for an S-groupoid $\mathfrak{X}$ immediately implies that (R-S) holds for the set-valued functor $U \times_{\mathfrak{X}} V$ on Aff/S. This implies Schlessinger's (H-1), (H-2) and (H-4) hold for $U \times_{\mathfrak{X}} V$ restricted to $Art_{k/\Lambda}$ at any k-point.
- The condition (S-2) (which corresponds to Schlessinger's (H-3) on $Art_{k/\Lambda}$) for the functor $U \times_{\mathfrak{X}} V$ amounts to a finiteness condition on infinitesimal automorphisms, addressed next.

Finiteness condition for $Aut_a(A[M])$ -(i)

- Let $\mathfrak{X}$ be a groupoid over $\mathcal{C}^{opp}$, let A be a ring in $\mathcal{C}$, and let $a \in Ob\,\mathfrak{X}(A)$. For any finite A-module M, let $a[M] \in Ob\,\mathfrak{X}(A[M])$ be the image of a under $A \hookrightarrow A[M]$ (that is, the pullback of a under the projection $\mathrm{Spec}\,A[M] \to \mathrm{Spec}\,A$). Note that we also have a closed embedding $\mathrm{Spec}\,A \hookrightarrow \mathrm{Spec}\,A[M]$ defined by the surjection $A[M] \to A : (a, m) \mapsto a$.
- We define $Aut_a(A[M]) \subset Aut(a[M])$ to be the subgroup consisting of all $\phi : a[M] \to a[M]$ in $\mathfrak{X}(A[M])$ such that $\phi|_A = \mathrm{id}_a$.
- Let $S_a(A[M])$ be the underlying subset of the group $Aut_a(A[M])$. If the stronger Rim-Schlessinger condition (R-S) is satisfied, then we have a natural bijection

  $$S_a(A[M \oplus M]) = S_a(A[M] \times_A A[M]) \stackrel{\sim}{\to} S_a(A[M]) \times S_a(A[M])$$

  which gives $S_a(A[M])$ the structure of an A-module.

Finiteness condition for $Aut_a(A[M])$ -(ii)

- If $u, v \in S_a(A[M])$, if $w \in S_a(A[M \oplus M])$ is the unique element which maps to (u, v), and $\beta : A[M \oplus M] \to A[M]$ is induced by the addition map $M \oplus M \to M$, then $g + h = \beta(w)$ by the definition of addition. Note that the definition of $g + h$ does not use the group structure on $Aut_a(A[M])$.
- The scalar multiplication in $S_a(A[M])$ by $\lambda \in A$ s induced by $A[M] \to A[M] : (a, m) \mapsto (a, \lambda m)$. This makes $S_a(A[M])$ a module over A.
- If $g, h \in Aut_a(A[M])$, and if $\pi_1, \pi_2 : A[M] \hookrightarrow A[M \oplus M]$ are the two inclusions $(a, m) \mapsto (a, m, 0)$ and $(a, m) \mapsto (a, 0, m)$, then $w = \pi_1(g) \circ \pi_1(h) \in Aut_a(A[M \oplus M])$ is an element of $S_a(A[M \oplus M])$ which maps to $(g, h) \in S_a(A[M]) \times S_a(A[M])$, so it is the unique such element ($\circ$ denotes the composition in $Aut_a(A[M \oplus M])$.
- Note that the composite $A[M] \xrightarrow{\pi_i} A[M \oplus M] \xrightarrow{\beta} A[M]$ is identity on $A[M]$ for $i = 1, 2$. Hence $\beta\pi_1(g) = g$ and $\beta\pi_2(h) = h$

Finiteness condition for $Aut_a(A[M])$ -(iii)

- It follows that $g + h = \beta(w) = \beta(\pi_1(g) \circ \pi_2(h)) g \circ h \in Aut_a(A[M])$, where $\circ$ denotes the group operation (composition of automorphisms) in $Aut_a(A[M \oplus M])$ or in $Aut_a(A[M])$.
- Thus, if (R-S) is satisfied by $\mathfrak{X}$, then each $Aut_a(A[M])$ is naturally an A-module, where the addition (sum of tangent vectors) equals the group multiplication (composition of automorphisms). In particular, the group $Aut_a(A[M])$ is necessarily commutative.
- We can directly demand the following, without asking for (R-S) to be satisfied:
- **Artin's finiteness condition for infinitesimal automorphisms** This is the requirement on the groupoid $\mathfrak{X}$ on $\mathcal{C}^{opp}$ that each $Aut_a(A[M])$ should be a **finite A-module**, where the addition is defined to be the composition of automorphisms, and the scalar multiplication is defined to be the map induced by $A[M] \to A[M] : (a, m) \mapsto (a, \lambda m)$.

Schlessinger's theorem

M. Schlessinger (1968): Functors of Artin rings.
Let $F : Art_{k/\Lambda} \to Sets$ be a functor, such that $F(k)$ is a singleton set. Then we have the following.

- **Theorem**: *F satisfies (S-1) and (S-2) (equivalently, (H-1), (H-2) and (H-3)), if and only if there exists a complete noetherian local Λ-algebra R with residue field k and a versal pro-family $(\xi_n)_{n\geq 0} \in \widehat{F}(R)$ for F over R. Moreover, F satisfies (H-1,2,3,4) if and only if a universal pro-family $(\xi_n)_{n\geq 0} \in \widehat{F}(R)$ exists for R.*
- Here, each $\xi_n \in F(R_n)$ where $R_n = R/\mathfrak{m}^{n+1}$, with $\xi_n = \xi_{n+1}|_{R_n}$. Also, once a versal family exists, we also have a miniversal family.
- The condition (H-4) says that if $A' \to A$ is surjective in $Art_{k/\Lambda}$ with $\ker(A' \to A) \cdot \mathfrak{m}_{A'} = 0$, then $F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$ is a bijection.
- Note that (S-1') is satisfied by any pro-representable functor F.

Sketch of proof -(i)

- We have functor $\Phi : FinMod/k \to Sets$ which sends $M \mapsto k[M] \mapsto F(k(M))$. Then $0 \mapsto F(k)$ which is a singleton set (terminal object), and
  $M \oplus N \mapsto F(k[M \oplus N]) = F(k[M] \times_k k[N]) \overset{(S-1b)}{=} F(k[M]) \times F(k[N])$
  so Φ preserves finite products, therefore (exercise!) lifts uniquely to $\Phi : FinMod/k \to Mod/k$.
- By (S-2) $F(k[M])$ is in $FinMod/k$. In particular,
- $T_F = F(k[\epsilon]/(\epsilon^2)) = \Phi(k)$ is a finite dim k-vector space.
- Φ is represented by T_F in the sense that $\Phi(M) = M \otimes_k T_F$ (exercise).

So far it was just linear algebra.

Sketch of proof -(ii)

Construction of a miniversal pro-family $(a_n) \in \widehat{F}(R)$ parametrized by a certain ring R in $\widehat{Art}_{k/\Lambda}$.

- Let P be the formal power series ring over k, which is the completion of the local ring at origin of the affine space T_F. Algebraically, $P = \widehat{\mathrm{Sym}_k(T_F^*)}$, completed at the maximal ideal generated by T_F^*. Let $\mathfrak{n} \subset P$ denote the maximal ideal.
- The ring $R = P/J$ is a quotient of P. The ideal J is constructed the intersection

$$\mathfrak{n}^2 = J_2 \supset J_3 \supset \ldots \supset \cap_{q=2}^{\infty} J_q = J$$

- Starting with $J_2 = \mathfrak{n}^2$, the ideals J_q are constructed iteratively, so that

$$J_q \supset J_{q+1} \supset \mathfrak{n}J_q \supset \mathfrak{n}^{q+1}$$

Sketch of proof -(iii)

- On $R_1 = P/J_2 = k[T_F^*]$, we have the universal first-order family given by
  $\mathrm{id}_{T_F} \in End(T_F) = T_F^* \otimes_k T_F = \Phi(T_F^*) = F(k[T_F^*]) = F(R_2)$.
  Let a_2 denote this family.
- We iteratively construct J_{q+1} and $a_q \in F(R_q)$ where $R_n = P/J_{n+1}$, such that J_{q+1} is the **unique smallest ideal** with $J_q \supset J_{q+1} \supset \mathfrak{n}J_q$ and such that $a_{q-1} \in F(P/J_q)$ admits a lift to $F(P/J_{q+1})$. We choose any lift $a_q|_{R_{q-1}} = a_{q-1}$.
- **Important**: While J_q's will turn out to be unique, the a_q will not necessarily be so.

Sketch of proof -(iv)

- A minimal ideal I such that (i) $J_q \supset I \supset \mathfrak{n}J_q$ and (ii) $a_{q-1} \in F(P/J_q)$ has at least one lift to $F(P/I)$ exists by descending chain condition on the Artin ring $P/\mathfrak{n}J_q$ (which is a quotient of $J/\mathfrak{n}^{q+1}$).
- If $I_1,\, I_2$ are two such ideals then $I_1 \cap I_2$ is again such an ideal, so such a minimal ideal is unique. (This verification uses a small trick, and also the hypothesis (H 1)).
- Now choose an **arbitrary** lift a_{q+1}.
- Let $I_q = J_q/J \subset P/J = R$. Let $\mathfrak{m} = \mathfrak{n}/J \subset R$ its maximal ideal. It is easy to check using Mittag-Leffler condition that $R = \lim_{\leftarrow} R/I_q$, and for any $n \geq 1$ there exists $q \geq n$ with $I_{n-1} \supset \mathfrak{m}^n \supset I_q$. Hence (a_q) defines an element of $\lim_{\leftarrow} F(R/\mathfrak{m}^n) = \widehat{F}(R)$.
- We will omit the verification that this pro-family $(a_n) \in \widehat{F}(R)$ is formally versal.

Sketch of proof -(v)

- If (H 4) is also satisfied, then the pro-family is universal. This is because for any small extension $B \to A$ in $Art_{k/\Lambda}$ with kernel I, the fibers of $F(B) \to F(A)$ are principal $T_F \otimes_k I$-sets, and the fibers of $h_R(B) \to h_R(A)$ are principal $T_R \otimes_k I$-sets. But $T_F = T_R$ by construction of R. So if the natural map $h_R(A) \to F(A)$ induced by (a_n) is a bijection, then the natural map $h_R(B) \to F(B)$ is again so because of the following commutative diagram (top row is $T_F \otimes_k I$-equivariant).

$$\begin{array}{ccc} h_R(B) & \to & F(B) \\ \downarrow & & \downarrow \\ h_R(A) & = & F(A) \end{array}$$

- Applying the above iteratively from q to $q+1$, it follows that (a_q) defines a bijection $h_R \to F$ on $Art_{k/\Lambda}$.

Grothendieck Existence Theorem

R complete noetherian local ring, X a proper scheme over R.
$R_n = R/\mathfrak{m}^{n+1}$. $X_n = X \otimes_R R_n$.
$X_0 \subset X_1 \subset \ldots$ are square-zero extensions.
E coherent $\mathcal{O} + X$-module. Each $E_n = E|_{X_n}$ is coherent on X_n. Let $u_n : E_n \to E_{n+1}|_{X_n}$ denote the induced isomorphisms.
Theorem: The functor $E \mapsto (E_n, u_n)_{n \geq 0}$ is an equivalence of categories.
Application: Effectiveness for Hilbert and quot functors (blackboard).

[Artin 1969] Existence of Algebraization

S excellent base, $F : Rings/S \to Sets$ functor. Then F is representable by a (separated) algebraic space of finite type over S if and only if:

- (Descent) F is an étale sheaf.
- (Finite type) F is locally of finite presentation.
- (Effectivity) F is effectively pro-representable.
- (Strong representability of diagonal) If U is finite type over S and $\xi, \eta \in F(U)$ then $\xi = \eta$ defines a (closed) subscheme of of U.
- (Openness of versality) If U is finite type over S and $\xi \in F(U)$ is formally étale at $P \in U$ then it is formally étale in a Zariski nbd of P in U.

Sketch of proof of sufficiency in unobstructed case (blackboard).

Obstruction theory -(i)

- An **obstruction theory** for $\mathfrak{X}$ means the following data:
- (i) For each infinitesimal extension $A \twoheadrightarrow A_0$ and object a in $\mathfrak{X}(A)$, we are given a functor $M \mapsto Obs_a(M)$ from the category of finite A_0-modules to itself.
- (ii) For each deformation situation $(A' \twoheadrightarrow A \overset{q_0}{\twoheadrightarrow} A_0,\, M)$ and object a in $\mathfrak{X}(A)$, we are given an element $obs_a(A') \in Obs_a(M)$ which is zero if and only if a has a lift to $\mathfrak{X}(A')$.
- This data should be functorial, and linear in (A_0, M).
- **Basic example**: $\mathcal{C} = Art_k$ the category of Artin k-algebras with residue field k. Let R be a complete noetherian local k-algebra with residue field k. Then $R = P/J$ where $P = k[[t_1, \ldots, t_n]]/J$ where $n = dim(\mathfrak{m}_R/\mathfrak{m}_R^2)$, and $J \subset \mathfrak{m}_P^2$ where $\mathfrak{m}_P = (t_1, \ldots, t_n)$. Functor $h_R : Art_k \to Sets$.
- Automorphisms of h_R are trivial. Tangent: $h_R(k[\epsilon]/(\epsilon^2) = (\mathfrak{m}_R/\mathfrak{m}_R^2)^*$.
- Obstruction theory: Put $Obs_a(M) = (J/\mathfrak{m}_P J)^* \otimes_k M$.

Obstruction theory -(ii)

- Given $a \in h_R(A)$, that is, $a : R \to A$, by arbitrarily lifting the images of t_i, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & J & \to P \to & R & \to 0 \\ & \downarrow g & \downarrow f & \downarrow a & \\ 0 \to & M & \to A' \to & A & \to 0 \end{array}$$

- As $f(\mathfrak{m}_P) \subset \mathfrak{m}_{A'}$, it follows that $g(\mathfrak{m}_P J) \subset \mathfrak{m}_{A'} M = 0$. Hence we get a map $J/\mathfrak{m}_P J \to M$, that is, an element

$$obs_a(A') \in (J/\mathfrak{m}_P J)^* \otimes_k M.$$

- Clearly, a lift $a' : R \to A'$ exists for a if and only if $obs_a(A') = 0$.
- The set of all lifts is a principal set under $(\mathfrak{m}_R/\mathfrak{m}_R^2)^* \otimes_k M$.

Artin's Representability Theorem 5.3 [1974] - (i)

Let S be a locally noetherian base which is excellent.
Let $\mathfrak{X}$ be a groupoid on $Aff/S = (Rings/S)^{opp}$. The following conditions are necessary and sufficient for $\mathfrak{X}$ to be a locally finite type and locally quasi-separated algebraic stack over S.

- **Descent condition**: The S-groupoid $\mathfrak{X}$ is a stack on Aff/S in the fppf topology.
- **Locally finite type**: The S-groupoid $\mathfrak{X}$ is limit preserving: for any filtered direct system of rings A_i in $Rings/S$, we have a natural equivalence
$$\varinjlim \mathfrak{X}(A_i) \to \mathfrak{X}(\varinjlim A_i)$$
This corresponds to being locally of finite type over S.
- (5.3.1) **Infinitesimal conditions**: The S-groupoid $\mathfrak{X}$ satisfies Rim-Schlessinger condition (R-S), $D_{a_0}(M) = \overline{\mathfrak{X}_{a_0}}(A_0[M])$ is a finite A_0-module (S-2), and $Aut_a(A[M])$ is a finite A-module.

Artin's representability theorem 5.3 [1974] - (ii)

- (5.3.2) **Effectivity**: For any complete local ring $(R, \mathfrak{m})$ over S such that $R/\mathfrak{m}$ is of finite-type over S, the functor

$$\mathfrak{X}(R) \to \lim_{\leftarrow} \mathfrak{X}(R/\mathfrak{m}^{n+1})$$

  is fully faithful, and its image is dense (where 'dense' means $\mathfrak{X}(R) \to \mathfrak{X}(R/\mathfrak{m}^{n+1})$ is essentially surjective for $n \gg 0$).
  (Fact: If $\mathfrak{X}$ is an algebraic stack then $\mathfrak{X}(R) \to \lim_{\leftarrow} \mathfrak{X}(R/\mathfrak{m}^{n+1})$ is actually an equivalence of categories. The above effectivity condition is milder.)
- (5.3.3) There exists an obstruction theory *Obs* for $\mathfrak{X}$, such that *Inf*, *D* and *Obs* satisfy the conditions (4.1).
- (5.3.4) **Local quasi-separatedness** If $a_0 \in \mathfrak{X}(A_0)$ is algebraic and ϕ is an automorphism of a_0 which induces the identity in $\mathfrak{X}(k)$ for a dense set of points $A_0 \to k$ of finite type, then $\phi = \mathrm{id}$ on a non-empty open subset of $\mathrm{Spec}(A_0)$.

Artin conditions Vs Schlessinger conditions

For a set-valued functor F on $Art_{k/\Lambda}^{opp}$, such that $F(k)$ is a singleton set, the following three conditions are equivalent:

- (1) F is pro-representable.
- (2) F satisfies (R-S) (same as (S-1') of Artin) and (S-2).
- (3) F satisfies Schlessinger (H-1), (H-2), (H-3), (H-4).

For a groupoid $\mathfrak{X}$ on $Art_{k/\Lambda}^{opp}$ such that $\mathfrak{X}(k)$ is equivalent to a singleton set and Artin (R-S) is satisfied, the following two conditions are equivalent:

- (1) Given a surjection $A' \to A$ in $Art_{k/\Lambda}$, for any object $a' \in \mathfrak{X}(A')$, the induced homomorphism of groups $Aut(a') \to Aut(a'|_A)$ is surjective.
- (2) The functor $\overline{\mathfrak{X}}$ satisfies Schlessinger (H-4).

Algebraization -(i)

- Let R be in $\widehat{Art}_{k/\Lambda}$, and let $(\xi_n, u_n) \in \widehat{\mathfrak{X}_a}(R)$ be a formal object. Here, $a = \xi_0$, and $u_n : \xi_n \xrightarrow{\sim} \xi_{n+1}|_{R_n}$ where $R_n = R/\mathfrak{m}_R^{n+1}$.
- **Question** Is the formal deformation **effective**, that is, does there exist (ξ, v_n) where $\xi \in \mathfrak{X}_a$ and $v_n : \xi_n \xrightarrow{\sim} \xi|_{R_n}$ compatible with the u_n?
- **Answer** Not always! But there is a theorem of Grothendieck which can often be used to get a positive answer.
- **Grothendieck existence theorem: special case** Let R be a complete noetherian local ring, let $X \to \mathrm{Spec}\, R$ be a proper morphism of schemes, and let $(E_n, u_n)_{n \geq 0}$ be coherent sheaves on $X_n = X \otimes_R R_n$ with isomorphisms $u_n : E_n \xrightarrow{\sim} E_{n+1}|_{X_n}$. Then there exists a coherent sheaf E on X and isomorphisms $v_n : E_n \xrightarrow{\sim} E|_{X_n}$ compatible with the u_n.
- For a modern treatment, see Illusie's article in 'FGA Explained'.

Algebraization -(ii)

- Let k be a field of finite type over S, let $\xi_0 \in Ob\,\mathfrak{X}(k)$, and let R be a noetherian complete local ring over S with residue field k, with an object $\xi \in \mathfrak{X}_{\xi_0}(R)$ which is smooth over $\overline{\mathfrak{X}_{\xi_0}}$.
- This gives a pro-object (ξ_n, u_n) where $\xi_n = \xi|_{R_n}$ and $u_n : \xi_n \stackrel{\sim}{\to} \xi_{n+1}|_{R_n}$ where $R_n = R/\mathfrak{m}_R^{n+1}$, which is formally versal over $\overline{\mathfrak{X}_{\xi_0}}$. But we have begun with an actual object $\xi \in \mathfrak{X}_{\xi_0}(R)$, that is, we have effectivity built into our hypothesis.
- We want a scheme U of finite type over S, a closed point $P_0 \in U$ with residue field k, and an object $\eta \in \mathfrak{X}(U)$ with an isomorphism $\xi_0 \to \eta|_{P_0}$, and an S-morphism $\mathcal{O}_{U,P_0} \to R$ which induces an isomorphism $\widehat{\mathcal{O}}_{U,P_0} \to R$, such that for each $n \geq 0$, η restricts to ξ_n under the composite $\mathcal{O}_{U,P_0} \to R \to R_n$.
- **Artin's theorem on algebraization**: The above is realizable if the S-groupoid $\mathfrak{X}$ is limit preserving and S is excellent. The chief ingredient is the Artin approximation theorem, which needs S to be excellent.

Algebraization -(iii)

Excellent rings are a class of noetherian commutative rings that are 'sufficiently well behaved' for doing algebraic geometry. The rings which one usually encounters in usual algebraic geometry are indeed excellent. The definition is technical – instead, we will give some examples:

- Complete noetherian local rings, in particular, all fields.
- Dedekind domains of characteristic 0, in particular, $\mathbb{Z}$.
- Convergent power series over $\mathbb{R}$ or $\mathbb{C}$ in finitely many variables.
- Any localization of an excellent ring.
- Finite type algebras over an excellent ring.

Algebraization -(iv)

- Proof in special case: when $S = \mathrm{Spec}\, k$ and $R = k[[t]]$ (blackboard).
- The general case is much harder: see [Artin 1969] *Algebraization of formal moduli -I*
- The openness of formal versality show that there exists an open neighbourhood V of P_0 in U such that $\eta|V$ is formally smooth over $\mathfrak{X}$.
- Starting with all possible k and $\xi_0 \in \mathfrak{X}(k)$, and taking disjoint union of the resulting schemes V, we get a smooth atlas for $\mathfrak{X}$.

Artin [1974] conditions on *Inf*, *Tan*, *Obs*: (4.1)(i)

Let $\mathfrak{X}$ be a limit-preserving groupoid on Aff/S. Suppose $\mathfrak{X}$ satisfies (S-1,2) and suppose we have an obstruction theory *Obs* for $\mathfrak{X}$. Following are Artin [1974] conditions (4.1) on *Inf*, *Tan*, *Obs*. Let A be of finite type over S, let $A_0 = A/\mathit{Nil}(A)$, let M be a finite A_0-module.

- (4.1)(i) **Compatibility with étale base-changes**: Let $A \to B$ be étale, and let $B_0 = B \otimes_A A_0$. Let $a \in \mathfrak{X}(A)$ (means a is an 'algebraic object'), and let $a_0 \in \mathfrak{X}(A_0)$, $b \in \mathfrak{X}(B)$ and $b_0 \in \mathfrak{X}(B_0)$ denote its various pullbacks. Then the natural maps below are isomorphisms:

$$\mathit{Inf}_{b_0}(M \otimes_{A_0} B_0) \stackrel{\sim}{\to} \mathit{Inf}_{a_0}(M) \otimes_{A_0} B_0,$$

$$D_{b_0}(M \otimes_{A_0} B_0) \stackrel{\sim}{\to} D_{a_0}(M) \otimes_{A_0} B_0, \text{ and}$$

$$\mathit{Obs}_{b_0}(M \otimes_{A_0} B_0) \stackrel{\sim}{\to} \mathit{Obs}_{a_0}(M) \otimes_{A_0} B_0.$$

Artin [1974] conditions on *Inf*, *Tan*, *Obs*: (4.1) (ii), (iii)

- (4.1)(ii) **Compatibility with completions**: *Inf* and *D* are compatible with completions at maximal ideals $\mathfrak{m} \subset A_0$:

$$Inf_{a_0}(M \otimes_{A_0} \widehat{A_0}) \xrightarrow{\sim} \varprojlim Inf_{a_0}(M/\mathfrak{m}^{n+1}), \text{ and}$$

$$D_{a_0}(M \otimes_{A_0} \widehat{A_0}) \xrightarrow{\sim} \varprojlim D_{a_0}(M/\mathfrak{m}^{n+1}).$$

- (4.1)(iii) **Constructibility**: There exists a dense open subset of the set of all points of finite type $\mathrm{Spec}\, A_0$ such that at any p in the subset the following natural maps are isomorphisms,

$$Inf_{a_0}(M) \otimes_{A_0} k(p) \xrightarrow{\sim} Inf_{a_0}(M \otimes_{A_0} k(p)) \text{ and}$$

$$D_{a_0}(M) \otimes_{A_0} k(p) \xrightarrow{\sim} D_{a_0}(M \otimes_{A_0} k(p))$$

  and the following natural map is injective.

$$Obs_a(M) \otimes_{A_0} k(p) \hookrightarrow Obs_a(M \otimes_{A_0} k(p))$$

Openness of versality - (i)

- A morphism of schemes $f : X \to Y$ is **formally smooth** if given any square-zero thickening $\mathrm{Spec}\, A \to \mathrm{Spec}\, A'$ of affine schemes over Y, any Y-morphism $\mathrm{Spec}\, A \to X$ prolongs to an Y-morphism $\mathrm{Spec}\, A' \to X$.
- Fact: f is a smooth morphism if and only if (i) f is locally of finite presentation and (ii) f is formally smooth.
- For a limit-preserving S-groupoid $\mathfrak{X}$, and R a ring of finite-type over S, an object $v \in \mathfrak{X}(R)$ ('algebraic element') is said to be **formally smooth** if if given any square-zero thickening $\mathrm{Spec}\, A \to \mathrm{Spec}\, A'$ of affine schemes over S, an S-morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$, and a lift $a' \in \mathfrak{X}(A')$ of $a = v|_A \in \mathfrak{X}(A)$, there exists an S-morphism $\mathrm{Spec}\, A' \to \mathrm{Spec}\, R$ and an isomorphism $a' \to v|_{A'}$ which restricts to identity on $\mathrm{Spec}\, A$.
- The algebraic element $v \in \mathfrak{X}(R)$ is said to be **formally versal at a point** $p \in \mathrm{Spec}\, R$ if the above holds whenever A and A' are Artin local rings with residue field $k(p)$.

Openness of versality - (ii)

[Artin 1974] If $\mathfrak{X}$ is a limit-preserving S-groupoid, with an obstruction theory, such that (4.1) holds, then the following important facts can be proved:

- **Proposition (4.2)** An algebraic element $v \in \mathfrak{X}(R)$ is formally smooth over $\mathfrak{X}$ if and only if it is formally versal at every point $p \in \mathrm{Spec}\, R$ of finite type.
- **Proposition (4.3)** Formal versality is stable under étale base change.
- **Theorem (4.4)** If an algebraic element $v \in \mathfrak{X}(R)$ is formally versal at a finite-type point $p \in \mathrm{Spec}\, R$, then p has an open nbd in which v is formally smooth. In particular, formal versality is an open condition: v is formally versal at each finite-type point in the nbd.

Misprints in hypothesis of Artin [1974] Theorem 5.3

In the statement of Theorem 5.3, there are two misprints:
The hypothesis (1) should include the demand that **(S-1')** should hold (the original text just says (S-1) should hold).
The hypothesis (2) should include the demand that the canonical functor $F(\widehat{A}) \to \lim_{\leftarrow} F(A/\mathfrak{m}^{n+1})$ is **fully faithful** (the original text just says it should be faithful).

Moreover, according to Hall and Rydh [2012], the S-stack $\mathfrak{X}$ should be assumed to be an **fppf stack** (not just an étale stack): if the stack is to be assumed to be just étale, then some other changes will be needed.

Artin [1974] Main Theorem (briefly restated)

Let S be an excellent scheme, and let $\mathfrak{X}$ be an fppf stack over S. Then $\mathfrak{X}$ is a finitely presented locally quasi-separated algebraic stack over S if and only if the following conditions are satisfied.

- **(1)** Infinitesimal conditions: (S-1') and (S-2) hold. If $a_0 \in \mathfrak{X}(A_0)$, for a reduced ring A_0 of finite type over S, and M is a finite A_0 module, then $Inf_{a_0}(A_0[M])$ is a finite A_0-module.
- **(2)** Effectivity holds: $\mathfrak{X}(R) \to \lim_{\leftarrow} \mathfrak{X}(R/\mathfrak{m}^{n+1})$ is fully faithful with dense image for R complete local.
- **(3)** An obstruction theory exists, and Aut-Tan-Obs satisfy (4.1).
- **(4)** Local quasi-separatedness: Any automorphism of an object over a finite-type reduced algebra which is identity on a dense set of finite-type points is identity on a non-empty open subset

Further progress beyond Artin [1974]

The following is a list (may not be exhaustive – my apologies!) of some major developments.

- H. Flenner (1981): Ein Kriterium fur die Offenheit der Versalitat.
- B Conrad and A de Jong (2002): Approximation of versal deformations.
- M Olsson (2006): Deformation theory of representable morphisms of alg stacks.
- J. M. Starr (2006): Artin's axioms, composition and moduli spaces,
- M Olsson (2007): Sheaves on Artin stacks.
- J. Wise (2011): Obstruction theories and virtual fundamental classes.
- J. Hall (2011): Openness of versality via coherent functors.
- J. Hall and D. Rydh (2012): Artin's criteria for algebraicity revisited.
- Multi-author effort (on-going): The Stacks Project.