Master class

Vortex equations and Hamiltonian GW invariants

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1. Introduction

The aim of this master class is to explain the definition and main properties of Hamiltonian Gromov–Witten (HGW) invariants of compact symplectic manifolds with a Hamiltonian action of S^1, following [20]. These invariants are defined by counting gauge equivalence classes of solutions to the vortex equations. We will define the invariants in the most general case, where the complex curve on which the vortex equations are considered is allowed to vary within the Deligne–Mumford moduli space. We will also comment briefly on other works on HGW invariants. Time (and energy) permitting, we will say a few words on Hitchin–Kobayashi correspondence.

We will try to be reasonably self-contained, giving when possible the necessary background. For references and some suggested reading to prepare the master class, see Section 4.

2. Schedule of the talks

	Tuesday 17	**Wednesday 18**	**Thursday 19**	**Friday 20**
10:00 - 10:45	Intro I	Cpctness. I	Fredholm II	Perturb. I
11:15 - 12:00	Intro II	Cpctness. II	Fredholm III	Perturb. II
14:00 - 14:45	Diag. class I	Cpctness. III	Eval. map I	Perturb. III
15:15 - 16:00	Diag. class II	Fredholm I	Eval. map II	Axioms

3. Contents of the lectures

3.1. **Introduction I.** Symplectic manifolds, Hamiltonian actions and symplectic quotients. Invariant almost complex structures. The vortex equations: twisted holomorphic maps (THM's). Gauge invariance. Equivariant cohomology. The moduli space (naive approach).

3.2. **Introduction II.** Statement of the theorems:

(1) Existence of biinvariant diagonal class.
(2) Existence of the invariants.
(3) Splitting axiom.
(4) Associativity of the Hamiltonian quantum product in equivariant cohomology.

(5) Relation to Chen-Ruan orbifold product.

3.3. **Biinvariant diagonal class I.** The Kirwan map and Kirwan surjectivity. Gradient flow lines. Compactification: broken gradient flow lines. The conflict between transversality and equivariance.

3.4. **Biinvariant diagonal class II.** Multivalued perturbations of the almost complex structure defined on solid torii. Perturbed chains of gradient segments. Definition of the biinvariant diagonal class. Proof of the main theorems: estimate of the dimension of the strata at infinity.

3.5. **Compactness I.** Stable curves and Deligne-Mumford moduli space. Canonical volume forms. Meromorphic connections. Critical and generic holonomies. THM's of bounded geometry on long cylinders: limit orbits. Definition of stable twisted holomorphic maps (c-STHM's).

3.6. **Compactness II.** Isomorphisms of c-STHM's. The automorphism group of a c-STHM and critical values of c. Fundamental class represented by a c-STHM. The Yang–Mills–Higgs functional. The main compactness theorem. Statement of the main local estimates: on a complex curve of bounded geometry, and on long cylinders. Proof of the main theorem using local estimates.

3.7. **Compactness III.** Proof of the local estimates.

3.8. **Fredholm I.** Sobolev spaces with weights on punctured complex curves. The deformation complex $\mathcal{D}^{\bullet}_{(A,\phi)}$ of a THM (A,ϕ) with punctures. The index of the deformation complex $\mathcal{D}^{\bullet}_{(A,\phi)}$. From the deformation complex to Fredholm operators.

3.9. **Fredholm II.** Perturbed Laplacian and Riemann–Roch for meromorphic connections. Example: flat meromorphic bundles on a punctured projective line. Linearization of vortex equations on line bundles.

3.10. **Fredholm III.** Deformation complex of perturbed chain of gradient segments. Deformation complex $\mathcal{D}^{\bullet}_{\mathcal{C}}$ of a c-STHM $\mathcal{C}$. Computing the index of $\mathcal{D}^{\bullet}_{\mathcal{C}}$. Orientability.

3.11. **Evaluation map I.** Poincaré (orbi)bundle over the moduli space of c-STHM's: local triviality. The approach of [16]: when the complex curve is fixed. Resolutions of orbifolds.

3.12. **Evaluation map II.** The Laplacian on complex curves converging to a nodal curve. The evaluation points.

3.13. **Perturbations I.** Perturbations of chains of gradient segments, perturbations on long cylinders near gradient segments, and perturbations on the locus with bounded geometry. Definition of canonical long necks in a way compatible with gluing.

3.14. **Perturbations II.** Perturbed twisted holomorphic cylinders.

3.15. **Perturbations III.** Definition of the invariants.

3.16. **Axioms.** Proof of the splitting axiom. Hamiltonian quantum product on equivariant cohomology. Associativity. Relation of Hamiltonian Quantum product to Chen–Ruan orbifold product. Hitchin–Kobayashi correspondence.

4. References of the material and some reading suggestions

For an introduction to symplectic manifolds, Hamiltonian actions and equivariant cohomology two very good references are [1, 14]. Lectures **Intro I, II** will be based on this material and on [16]. Lectures **Diag. class I, II** will be based on [17]. Lectures **Cpctness. I, II, III**, will be based on [19] (some acquaintance with Gromov's compactness theorem for stable (pseudo)holomorphic maps would be helpful in these lectures; two very good references for this, amongst many others which are also very good, are [6, 13]). Lectures **Fredholm I, II, III**, **Eval. map I, II**, **Perturb. I, II, III**, and **Axioms** will be based on [20] (which will hopefully be available soon). A useful reference for Hitchin–Kobayashi correspondence in the setting relevant to HGW invariants might be [15, 8]; the paper [18] treats the finite dimensional case.

To read about other approaches to HGW invariants (with different degrees of generality) and different results related to them, the reader might wish to have a look at [2, 3, 4, 7, 9, 10, 11, 12, 21, 22, 23, 24, 25, 26]. The paper [5] makes some steps towards an algebraic definition of HGW invariants.

References

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