LECTURES ON ALGEBRAIC CATEGORIFICATION

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Contents

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1. Basics: decategorification and categorification

1.1. **The idea of categorification.** The term "categorification" was introduced by Louis Crane in [Cr] and the idea originates from the earlier joint work [CF] with Igor Frenkel. The term refers to the process of replacing set-theoretic notions by the corresponding category-theoretic analogues as shown in the following table:

Set Theory	Category Theory
set	category
element	object
relation between elements	morphism of objects
function	functor
relation between functions	natural transformation of functors

The general idea (or hope) is that, replacing a "simpler" object by something "more complicated", one gets a bonus in the form of some extra structure which may be used to study the original object. A priori there are no explicit rules how to categorify some object and the answer might depend on what kind of extra structure and properties one expects.

Example 1.1. The category $\mathscr{F}\mathscr{S}$ of finite sets may be considered as a categorification of the semi-ring $(\mathbb{N}_0, +, \cdot)$ of non-negative integers. In this picture addition is categorified via the disjoint union and multiplication via the Cartesian product. Note that the categorified operations satisfy commutativity, associativity and distributivity laws only up to a natural isomorphism.

In these lectures we will deal with some rather special categorifications of algebraic objects (which are quite different from the above example). There exist many others, even for the same objects. Our categorifications are usually motivated by the naturality of their constructions and various applications.

It is always easier to "forget" information than to "make it up". Therefore it is much more natural to start the study of categorification with the study of the opposite process of forgetting information, called *decategorification*. One of the most natural classical ways to "forget" the categorical information encoded in a category is to consider the corresponding *Grothendieck group*.

1.2. **Grothendieck group.** Originally, the Grothendieck group is defined for a commutative monoid and provides the universal way of making that monoid into an abelian group. Let $M = (M, +, 0)$ be a commutative monoid. The *Grothendieck group* of M is a pair (G, φ), where G is a commutative group and $\varphi : M \to G$ is a homomorphism of monoids, such that for every monoid homomorphism $\psi : M \to A$, where A is a commutative group, there is a unique group homomorphism $\Psi : G \to A$ making the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & G \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\Psi} & \\ & A & \end{array}$$

In the language of category theory, the functor that sends a commutative monoid M to its Grothendieck group G is left adjoint to the forgetful functor from the category of abelian groups to the category of commutative monoids. As usual, uniqueness of the Grothendieck group (up to isomorphism) follows directly from the universal property. Existence is guaranteed by the following construction:

Consider the set $G = M \times M/\sim$, where $(m, n) \sim (x, y)$ if and only if $m+y+s = n + x + s$ for some $s \in M$.

Lemma 1.2. (*a*) *The relation $\sim$ is a congruence on the monoid $M\times M$ (i.e. $a \sim b$ implies $ac \sim bc$ and $ca \sim cb$ for all $a, b, c \in M \times M$) and the quotient G is a commutative group (the identity element of G is $\overline{(0, 0)}$; and the inverse of $\overline{(m, n)}$ is $\overline{(n, m)}$).*
(*b*) *The map $\varphi : M \to G$ defined via $\varphi(m) = \overline{(m, 0)}$ is a homomorphism of monoids.*
(*c*) *The pair (G, φ) is a Grothendieck group of M.*

This idea of the Grothendieck group can be easily generalized to the situation of categories with some additional structures. The most classical example is the Grothendieck group of an abelian category. Let $\mathscr{A}$ be an abelian category. Then the *Grothendieck group* $[\mathscr{A}] = \mathrm{K}_0(\mathscr{A})$ of $\mathscr{A}$ is defined as the quotient of the free abelian group generated by $[X]$, where $X \in \mathscr{A}$, modulo the relation $[Y] = [X]+[Z]$ for every exact sequence

$$0 \to X \to Y \to Z \to 0 \tag{1.1}$$

in $\mathscr{A}$. This comes together with the natural map $[\cdot] : \mathscr{A} \to [\mathscr{A}]$ which maps $M \in \mathscr{A}$ to the class $[M]$ of M in $[\mathscr{A}]$. The group $[\mathscr{A}]$ has the following natural universal property: fore every abelian group A and for every *additive* function $\chi : \mathscr{A} \to A$ (i.e. a function such that $\chi(Y) = \chi(X) + \chi(Z)$ for any exact sequence (1.1)) there is a unique group homomorphism $\overline{\chi} : [\mathscr{A}] \to A$ making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{[\cdot]} & [\mathscr{A}] \\ & {\scriptstyle\chi}\searrow \quad \swarrow{\scriptstyle\overline{\chi}} & \\ & A & \end{array}$$

The Grothendieck group of $\mathscr{A}$ is the "easiest" way to make $\mathscr{A}$ into just an abelian group. In some classical cases the group $[\mathscr{A}]$ admits a very natural description:

Example 1.3. Let $\Bbbk$ be a field and $\mathscr{A} = A$-mod the category of finite-dimensional (left) modules over some finite dimensional $\Bbbk$-algebra A. As every A-module has a composition series, the group $[\mathscr{A}]$ is isomorphic to the free abelian group with the basis given by classes of simple A-modules.

Similarly one defines the notion of a Grothendieck group for additive and triangulated categories. Let $\mathscr{A}$ be an additive category with biproduct $\oplus$. Then the *split Grothendieck group* $[\mathscr{A}]_\oplus$ of $\mathscr{A}$ is defined as the quotient of the free abelian group generated by $[X]$, where $X \in \mathscr{A}$, modulo the relations $[Y] = [X] + [Z]$ whenever $Y \cong X \oplus Z$. Note that any abelian category is additive, however, if $\mathscr{A}$ is abelian, then the group $[\mathscr{A}]_\oplus$ can be bigger than $[\mathscr{A}]$ if there are exact sequences of the form (1.1) which do not split.

Let $\mathscr{C}$ be a triangulated category. Then the *Grothendieck group* $[\mathscr{C}]$ of $\mathscr{C}$ is defined as the quotient of the free abelian group generated by $[X]$, where $X \in \mathscr{C}$, modulo the relations $[Y] = [X] + [Z]$ for every distinguished triangle

$$X \to Y \to Z \to X[1].$$

Again, a triangulated category is always additive, but $[\mathscr{C}]_\oplus$ is usually bigger than $[\mathscr{C}]$ by the same arguments as for abelian categories.

Let $\Bbbk$ be a field and A a finite dimensional $\Bbbk$-algebra. Then we have two naturally defined triangulated categories associated with A-mod: the bounded derived category $\mathcal{D}^b(A)$ and its subcategory $\mathcal{P}(A)$ of perfect complexes (i.e. complexes, quasi-isomorphic to finite complexes of A-projectives). For projective P the map

$$\begin{array}{ccc} [\mathcal{P}(A)] & \xrightarrow{\varphi} & [A\text{-mod}] \\ {[P]} & \mapsto & [P] \end{array}$$

is a group homomorphism.

Example 1.4. If A has finite global dimension, then $\mathcal{D}^b(A) \cong \mathcal{P}(A)$ and the map φ is an isomorphism. This means that in this case the group $[A\text{-mod}]$ has another distinguished basis, namely the one corresponding to isomorphism classes of indecomposable projective A-modules.

Example 1.5. Consider the algebra $D = \mathbb{C}[x]/(x^2)$ of *dual numbers*. This algebra has a unique simple module $L := \mathbb{C}$ (which is annihilated by x). The projective cover of L is isomorphic to the left regular module $P := {}_D D$. The module P has length 2. Therefore the group $[D\text{-mod}]$ is the free abelian group with basis $[L]$. The group $[\mathcal{P}(D)]$ is the free abelian group with basis $[P]$ and $\varphi([P]) = 2[L]$. This means that $[\mathcal{P}(D)]$ is a proper subgroup of $[D\text{-mod}]$.

1.3. **Decategorification.**

Definition 1.6. Let $\mathscr{C}$ be an abelian or triangulated, respectively additive, category. Then the *decategorification* of $\mathscr{C}$ is the abelian group $[\mathscr{C}]$, resp. $[\mathscr{C}]_\oplus$.

In what follows, objects which we would like to categorify will usually be algebras over some base ring (field). Hence we now have to extend the notion of decategorification to allow base rings. This is done in the usual way (see [MS4, Section 2]) as follows: Let $\mathbb{F}$ be a commutative ring with 1.

Definition 1.7. Let $\mathscr{C}$ be an abelian or triangulated, respectively additive, category. Then the $\mathbb{F}$-*decategorification* of $\mathscr{C}$ is the $\mathbb{F}$-module $[\mathscr{C}]^{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{Z}} [\mathscr{C}]$ (resp. $[\mathscr{C}]^{\mathbb{F}}_\oplus := \mathbb{F} \otimes_{\mathbb{Z}} [\mathscr{C}]_\oplus$).

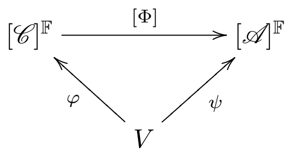
The element $1 \otimes [M]$ of some $\mathbb{F}$-decategorification will be denoted by $[M]$ for simplicity. We have $[\mathscr{C}] = [\mathscr{C}]^{\mathbb{Z}}$ and $[\mathscr{C}]_\oplus = [\mathscr{C}]^{\mathbb{Z}}_\oplus$.

1.4. **(Pre)categorification of an $\mathbb{F}$-module.**

Definition 1.8. Let V be an $\mathbb{F}$-module. An $\mathbb{F}$-*precategorification* $(\mathscr{C}, \varphi)$ of V is an abelian (resp. triangulated or additive) category $\mathscr{C}$ with a fixed monomorphism φ from V to the $\mathbb{F}$-decategorification of $\mathscr{C}$. If φ is an isomorphism, then $(\mathscr{C}, \varphi)$ is called an $\mathbb{F}$-*categorification* of V.

Whereas the decategorification of a category is uniquely defined, there are usually many different categorifications of an $\mathbb{F}$-module V. For example, in case $\mathbb{F} = \mathbb{Z}$ we can consider the category A-mod for any $\Bbbk$-algebra A having exactly n simple modules and realize A-mod as a categorification of the free module $V = \mathbb{Z}^n$. In particular, V has the *trivial* categorification given by a semisimple category of the appropriate size, for example by $\mathbb{C}^n$-mod.

Definition 1.9. Let V be an $\mathbb{F}$-module. Let further $(\mathscr{C}, \varphi)$ and $(\mathscr{A}, \psi)$ be two $\mathbb{F}$-(pre)categorifications of V via abelian (resp. triangulated or additive) categories. An exact (resp. triangular or additive) functor $\Phi : \mathscr{C} \to \mathscr{A}$ is called a *morphism of categorifications* provided that the following diagram commutes:



where $[\Phi]$ denotes the $\mathbb{F}$-linear transformation induced by Φ.

Definition 1.9 turns all $\mathbb{F}$-(pre)categorifications of V into a category. In what follows we will usually categorify $\mathbb{F}$-modules using module categories for finite dimensional $\Bbbk$-algebras. We note that extending scalars without changing the category $\mathscr{C}$ may turn an $\mathbb{F}$-precategorification of V into an $\mathbb{F}'$-categorification of $\mathbb{F}' \otimes_{\mathbb{F}} V$.

Example 1.10. Consider the algebra D of dual numbers (see Example 1.5). Then there is a unique monomorphism $\varphi : \mathbb{Z} \to [D\text{-mod}]$ such that $\varphi(1) = [D]$. The homomorphism φ is not surjective, however, it induces an isomorphism after tensoring over $\mathbb{Q}$, that is $\overline{\varphi} : \mathbb{Q} \xrightarrow{\sim} [D\text{-mod}]^{\mathbb{Q}}$. Hence $(D\text{-mod}, \varphi)$ is a pre-categorification of $\mathbb{Z}$ while $(D\text{-mod}, \overline{\varphi})$ is a $\mathbb{Q}$-categorification of $\mathbb{Q} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$.

The last example to some extend explains the necessity of the notion of precategorification. We will usually study categorifications of various modules. Module structures will be categorified using functorial actions, say by exact functors. Such functors are completely determined by their action on the additive category of projective modules. This means that in most cases the natural “basis” for categorification in the one given by indecomposable projectives. As we saw in Example 1.5, isomorphism classes of indecomposable projectives do not have to form a basis of the decategorification.

1.5. **Graded setup.** By *graded* we will always mean $\mathbb{Z}$-graded. Let R be a graded ring. Consider the category R-gMod of all graded R-modules and denote by $\langle 1 \rangle$ the shift of grading autoequivalence of R-gMod normalized as follows: for a graded module $M = \oplus_{i \in \mathbb{Z}} M_i$ we have $(M\langle 1 \rangle)_j = M_{j+1}$. Assume that $\mathscr{C}$ is a category of graded R-modules closed under $\langle \pm 1 \rangle$ (for example the abelian category R-gMod or the additive category of graded projective modules or the triangulated derived category of graded modules). Then the group $[\mathscr{C}]$ (resp. $[\mathscr{C}]_{\oplus}$) becomes a $\mathbb{Z}[v, v^{-1}]$-module via $v^i[M] = [M\langle -i \rangle]$ for any $M \in \mathscr{C}$, $i \in \mathbb{Z}$.

To extend the notion of decategorification to a category of graded modules (or complexes of graded modules), let $\mathbb{F}$ be a unitary commutative ring and $\iota : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}$ be a fixed homomorphism of unitary rings. Then ι defines on $\mathbb{F}$ the structure of a (right) $\mathbb{Z}[v, v^{-1}]$-module.

Definition 1.11. The *ι-decategorification* of $\mathscr{C}$ is the $\mathbb{F}$-module

$$[\mathscr{C}]^{(\mathbb{F},\iota)} := \mathbb{F} \otimes_{\mathbb{Z}[v,v^{-1}]} [\mathscr{C}] \quad (\text{resp. } [\mathscr{C}]_{\oplus}^{(\mathbb{F},\iota)} := \mathbb{F} \otimes_{\mathbb{Z}[v,v^{-1}]} [\mathscr{C}]_{\oplus}).$$

In most of our examples the homomorphism $\iota : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}$ will be the obvious canonical inclusion. In such cases we will omit ι in the notation. We have

$$[\mathscr{C}] = [\mathscr{C}]^{(\mathbb{Z}[v,v^{-1}],\mathrm{id})}, \quad [\mathscr{C}]_{\oplus} = [\mathscr{C}]_{\oplus}^{(\mathbb{Z}[v,v^{-1}],\mathrm{id})}.$$

Definition 1.12. Let V be an $\mathbb{F}$-module. A *ι-precategorification* $(\mathscr{C}, \varphi)$ of V is an abelian or triangulated, respectively additive, category $\mathscr{C}$ with a fixed free action of $\mathbb{Z}$ and a fixed monomorphism φ from V to the $(\mathbb{F}, \iota)$-decategorification of $\mathscr{C}$. If φ is an isomorphism, $(\mathscr{C}, \varphi)$ is called a *ι-categorification* of V.

Example 1.13. The algebra D of dual numbers is naturally graded with x being of degree 2 (this is motivated by the realization of D as the cohomology ring of a flag variety). Let $\mathscr{C} = D$-gmod. Then $[\mathscr{C}] \cong \mathbb{Z}[v, v^{-1}]$ as a $\mathbb{Z}[v, v^{-1}]$-module, hence the graded category $\mathscr{C}$ is a $(\mathbb{Z}[v, v^{-1}], \mathrm{id})$-categorification of $\mathbb{Z}[v, v^{-1}]$.

1.6. **Some constructions.** As already mentioned before, for any $\mathbb{F}$ we have the *trivial* categorification of the free module $\mathbb{F}^n$ given by $\Bbbk^n$-mod and the isomorphism $\varphi : \mathbb{F}^n \to [\Bbbk^n\text{-mod}]^{\mathbb{F}}$ which maps the usual basis of $\mathbb{F}^n$ to the basis of $[\Bbbk^n\text{-mod}]^{\mathbb{F}}$ given by isomorphism classes of simple modules.

If A-mod categorifies some $\mathbb{F}^k$ and B-mod categorifies $\mathbb{F}^n$ for some finite dimensional $\Bbbk$-algebras A and B, then $A \oplus B$-mod categorifies $\mathbb{F}^k \oplus \mathbb{F}^n$. This follows from the fact that every simple $A \oplus B$-modules is either a simple A-module or a simple B-module.

If A-mod categorifies $\mathbb{F}^k$ and B-mod categorifies $\mathbb{F}^n$ for some finite dimensional $\Bbbk$-algebras A and B, then $A \otimes_{\Bbbk} B$-mod categorifies $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}^n$. This follows from the fact that simple $A \otimes_{\Bbbk} B$-modules are of the form $L \otimes_{\Bbbk} N$, where L is a simple A-module and N is a simple B-module.

Let $\mathscr{A} = A$-mod categorify $\mathbb{F}^k$ such that the natural basis of $\mathbb{F}^k$ is given by the isomorphism classes of simple A-modules. Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{A}$ (i.e. for any exact sequence (1.1) in $\mathscr{A}$ we have $Y \in \mathscr{C}$ if and only if $X, Z \in \mathscr{C}$). Then there is an idempotent $e \in A$ such that $\mathscr{C}$ is the category of all modules annihilated by e. Thus $\mathscr{C}$ is equivalent to B-mod, where $B = A/AeA$. The group $[\mathscr{C}]$ is a subgroup of $[\mathscr{A}]$ spanned by all simple A-modules belonging to $\mathscr{C}$. Hence $\mathscr{C}$ categorifies the corresponding direct summand V of $\mathbb{F}^k$. We also have the associated abelian quotient category $\mathscr{A}/\mathscr{C}$ which has the same objects as $\mathscr{A}$ and morphisms given by

$$\mathscr{A}/\mathscr{C}(X, Y) := \varinjlim \mathscr{A}(X', Y/Y'),$$

where the limit is taken over all $X' \subset X$ and $Y' \subset Y$ such that $X/X', Y' \in \mathscr{C}$. The category $\mathscr{A}/\mathscr{C}$ is equivalent to C-mod, where $C = eAe$ (see for example [AM, Section 9] for details). It follows that $\mathscr{A}/\mathscr{C}$ categorifies the quotient $\mathbb{F}^k/V$, which is also isomorphic to the direct complement of V in $\mathbb{F}^k$.

2. Basics: from categorification of linear maps to 2-categories

The aim of the section is to discuss various approaches to categorification of algebras and modules. An important common feature is that any such approach categorifies linear maps as a special case.

2.1. **Categorification of linear maps.** Let V, W be $\mathbb{F}$-modules and $f : V \to W$ be a homomorphism. Assume that $(\mathscr{A}, \varphi)$ and $(\mathscr{C}, \psi)$ are abelian (resp. additive, triangulated) $\mathbb{F}$-categorifications of V and W, respectively.

Definition 2.1. An $\mathbb{F}$-*categorification* of f is an exact (resp. additive, triangular) functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ such that $[\mathrm{F}] \circ \varphi = \psi \circ f$, where $[\mathrm{F}] : [\mathscr{A}]^{\mathbb{F}}_{(\oplus)} \to [\mathscr{C}]^{\mathbb{F}}_{(\oplus)}$ denotes the induced homomorphism. In other words, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \psi} \\ [\mathscr{A}]^{\mathbb{F}}_{(\oplus)} & \xrightarrow{\quad [\mathrm{F}] \quad} & [\mathscr{C}]^{\mathbb{F}}_{(\oplus)} \end{array}$$

For example, the identity functor is a categorification of the identity morphism; the zero functor is the (unique) categorification of the zero morphism.

2.2. **Naïve categorification.** Let A be an (associative) $\mathbb{F}$-algebra with a fixed generating system $\mathtt{A} = \{a_i : i \in I\}$. Given an A-module M, every a_i defines a linear transformation a_i^M of M.

Definition 2.2. A *naïve* $\mathbb{F}$*-categorification* of M is a tuple $(\mathscr{A}, \varphi, \{\mathrm{F}_i : i \in I\})$, where $(\mathscr{A}, \varphi)$ is an $\mathbb{F}$-categorification of M and for every $i \in I$ the functor F_i is an $\mathbb{F}$-categorification of a_i^M.

There are several natural ways to define morphisms of naïve categorifications. Let A and $\mathtt{A}$ be as above, M and N be two A-modules and $(\mathscr{A}, \varphi, \{\mathrm{F}_i : i \in I\})$, $(\mathscr{C}, \psi, \{\mathrm{G}_i : i \in I\})$ be naïve categorifications of M and N, respectively. In what follows dealing with different categorifications we always assume that they have the same type (i.e. either they all are abelian or additive or triangulated). By a *structural* functor we will mean an exact functor between abelian categories, an additive functor between additive categories and a triangular functor between triangulated categories.

Definition 2.3. A *naïve morphism* of categorifications from $(\mathscr{A}, \varphi, \{\mathrm{F}_i : i \in I\})$ to $(\mathscr{C}, \psi, \{\mathrm{G}_i : i \in I\})$ is a structural functor $\Phi : \mathscr{A} \to \mathscr{C}$ such that for every $i \in I$ the following digram commutes:

$$\begin{array}{ccc}
[\mathscr{A}]^{\mathbb{F}} & \xrightarrow{[\mathrm{F}_i]} & [\mathscr{A}]^{\mathbb{F}} \\
{\scriptstyle [\Phi]}\downarrow & & \downarrow{\scriptstyle [\Phi]} \\
[\mathscr{C}]^{\mathbb{F}} & \xrightarrow{[\mathrm{G}_i]} & [\mathscr{C}]^{\mathbb{F}},
\end{array}$$

where $[\Phi]$ denotes the morphism, induced by Φ.

Definition 2.4. A *weak morphism* from $(\mathscr{A}, \varphi, \{\mathrm{F}_i : i \in I\})$ to $(\mathscr{C}, \psi, \{\mathrm{G}_i : i \in I\})$ is a structural functor $\Phi : \mathscr{A} \to \mathscr{C}$ such that for every $i \in I$ the following digram commutes (up to isomorphism of functors):

$$\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\mathrm{F}_i} & \mathscr{A} \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\mathscr{C} & \xrightarrow{\mathrm{G}_i} & \mathscr{C}
\end{array}$$

Definition 2.5. A *strict morphism* from $(\mathscr{A}, \varphi, \{\mathrm{F}_i : i \in I\})$ to $(\mathscr{C}, \psi, \{\mathrm{G}_i : i \in I\})$ is a structural functor $\Phi : \mathscr{A} \to \mathscr{C}$ such that for every $i \in I$ the following digram commutes strictly:

$$\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\mathrm{F}_i} & \mathscr{A} \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\mathscr{C} & \xrightarrow{\mathrm{G}_i} & \mathscr{C}
\end{array}$$

Definitions 2.3–2.5 give rise to the *naïve* (resp. *weak* or *strict*) category of naïve categorifications of A-modules (with respect to the basis $\mathtt{A}$). These categories are in the natural way (not full) subcategories of each other.

Example 2.6. Consider the complex group algebra $\mathbb{C}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$. For $\lambda \vdash n$ let S_λ be the corresponding Specht module and d_λ be its dimension (the number of standard Young tableaux of shape λ). Choose in $\mathbb{C}[\mathbb{S}_n]$ the standard generating system $\mathtt{A}$ consisting of transpositions $s_i = (i, i+1)$, $i = 1, 2, \dots, n-1$. In

S_λ choose the basis consisting of standard polytabloids (see e.g. [Sa, Chapter 2]). Then the action of every s_i in this basis is given by some matrix $M_i = (m^i_{st})_{s,t=1}^{d_\lambda}$ with integral coefficients. Categorify S_λ via $\mathcal{D}^b(\mathbb{C}^{d_\lambda})$, such that the basis of standard polytabloids corresponds to the usual basis of $[\mathcal{D}^b(\mathbb{C}^{d_\lambda})]$ given by simple modules. Categorify the action of every s_i using the appropriate direct sum, given by the corresponding coefficient in M_i, of the identity functors (shifted by 1 in homological position in the case of negative coefficients). We obtain a (trivial) naïve categorification of the Specht module S_λ.

Similarly to Example 2.6 one can construct trivial naïve categorifications for any module with a fixed basis in which the action of generators has integral coefficients. Instead of $\mathbb{C}$-mod one can also use the category of modules over any local algebra. We refer the reader to [Ma7, Chapter 7] for more details.

2.3. **Weak categorification.** To define the naïve categorification of an A-module M we simply required that the functor F_i categorifies the action of a_i^M only *numerically*, that is only on the level of the Grothendieck group. This is an extremely weak requirement so it is natural to expect that there should exist lots of different categorifications of M and that it should be almost impossible to classify, study and compare them in the general case. To make the classification problem more realistic we should impose some extra conditions. To see what kinds of conditions we may consider, we have to analyze what kind of structure we (usually) have.

The most important piece of information which we (intentionally) neglected up to this point is that A is an algebra and hence elements of A can be multiplied. As we categorify the action of the elements of A via functors, it is natural to expect that the multiplication in A should be categorified as the composition of functors. As we have already fixed a generating system $\mathtt{A}$ in A, we can consider some presentation of A (or the corresponding image with respect to the action on M) relative to this generating system. In other words, the generators a_i's could satisfy some relations. So, we can try to look for functorial interpretations of such relations. Here is a list of some natural ways to do this:

- equalities can be interpreted as isomorphisms of functors;
- addition in A can be interpreted as direct sum of functors;
- for triangulated categorifications one could interpret the negative coefficient -1 as the shift by 1 in homological position (see Example 2.6), in particular, subtraction in A can be sometimes interpreted via taking cone in the derived category;
- negative coefficients could be made positive by moving the corresponding terms to the other side of an equality.

Another quite common feature is that the algebra A we are working with usually comes equipped with an anti-involution $*$. The most natural way for the functorial interpretation of an anti-involution is via (bi)adjoint functors. Of course one has to emphasize that none of the above interpretations is absolutely canonical. Still, we might give the following loose definition from [MS4] (from now on, if $\mathbb{F}$ is fixed, we will omit it in our notation for simplicity):

Definition 2.7. A naïve categorification $(\mathscr{A}, \varphi, \{\mathrm{F}_i : i \in I\})$ of an A-module M is called a *weak categorification* if it satisfies the conditions given by some chosen interpretations of defining relations and eventual anti-involution for A.

In what follows for a functor F we will denote by F^* the biadjoint of F (if it exists).

Example 2.8. Let $A = \mathbb{C}[a]/(a^2 - 2a)$ and $\mathtt{A} = \{a\}$. Let further $M = \mathbb{C}$ be the A-module with the action $a \cdot 1 = 0$ and $N = \mathbb{C}$ the A-module with the action

$a \cdot 1 = 2$. Consider $\mathscr{C} = \mathbb{C}$-mod, $\mathrm{F} = 0$ and $\mathrm{G} = \mathrm{Id}_{\mathscr{C}} \oplus \mathrm{Id}_{\mathscr{C}}$. Define $\varphi : M \to [\mathscr{C}]$ and $\psi : N \to [\mathscr{C}]$ by sending 1 to $[\mathbb{C}]$ (the class of the simple $\mathbb{C}$-module). The algebra A has the $\mathbb{C}$-linear involution $*$ defined via $a^* = a$. We have both $\mathrm{F}^* = \mathrm{F}$ and $\mathrm{G}^* = \mathrm{G}$. We interpret $a^2 - 2a = 0$ as $a \cdot a = a + a$. We have $\mathrm{F} \circ \mathrm{F} \cong \mathrm{F} \oplus \mathrm{F}$ and $\mathrm{G} \circ \mathrm{G} \cong \mathrm{G} \oplus \mathrm{G}$. Hence $(\mathscr{C}, \varphi, \mathrm{F})$ and $(\mathscr{C}, \psi, \mathrm{G})$ are weak categorifications of M and N, respectively. Note that $A \cong \mathbb{C}[\mathbb{S}_2]$ and that under this identification the modules M and N become the sign and the trivial $\mathbb{C}[\mathbb{S}_2]$-modules, respectively.

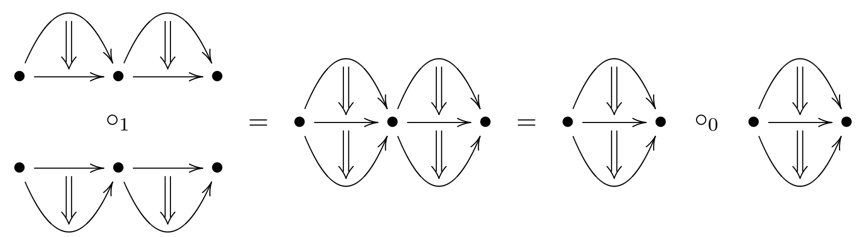
2.4. **2-categories.** Following our description above we now can summarize that to categorify the action of some algebra A on some module M we would like to "lift" this action to a functorial "action" of A on some category $\mathscr{C}$. So, the image of our "lift" should be a "nice" subcategory of the category of endofunctors on $\mathscr{C}$. This latter category has an extra structure, which we already tried to take into account in the previous subsections, namely, we can compose endofunctors. This is a special case of the structure known as a 2-*category*.

Definition 2.9. A 2-category is a category enriched over the category of categories.

This means that if $\mathscr{C}$ is a 2-category, then for any $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ the morphisms $\mathscr{C}(\mathtt{i}, \mathtt{j})$ form a category, its objects are called 1-*morphisms* and its morphisms are called 2-morphisms. Composition $\circ_0 : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ is called *horizontal* composition and is strictly associative and unital. Composition $\circ_1$ of 2-morphisms inside $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called *vertical* composition and is strictly associative and unital. We have the following *interchange law* for any composable 2-morphisms $\alpha, \beta, \gamma, \delta$:

$$(\alpha \circ_0 \beta) \circ_1 (\gamma \circ_0 \delta) = (\alpha \circ_1 \gamma) \circ_0 (\beta \circ_1 \delta)$$

This is usually depicted as follows: [1]



There is a weaker notion of a *bicategory*, in which, in particular, composition of 1-morphisms is only required to be associative up to a 2-isomorphism. There is a natural extension of the notion of categorical equivalence to bicategories, called *biequivalence*. We will, however, always work with 2-categories, which is possible thanks to the following statement:

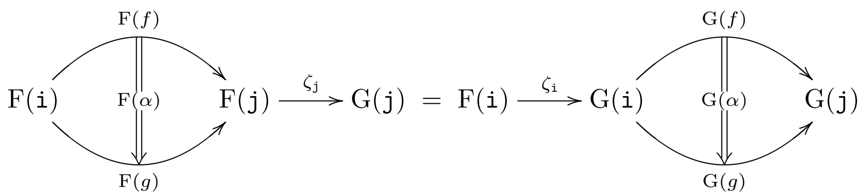
Theorem 2.10 ([MP, Le])**.** *Every bicategory is biequivalent to a* 2*-category.*

A typical example of a 2-category is the category of functors on some category. It has one object, its 1-morphisms are functors, and its 2-morphisms are natural transformations of functors. One can alternatively describe this using the notion of a (strict) *tensor category*, which is equivalent to the notion of a 2-category with one object.

A 2-functor $\mathrm{F} : \mathscr{A} \to \mathscr{C}$ between two 2-categories is a triple of functions sending objects, 1-morphisms and 2-morphisms of $\mathscr{A}$ to items of the same type in $\mathscr{C}$ such that it preserves (strictly) all the categorical structures. If $\mathrm{G} : \mathscr{A} \to \mathscr{C}$ is another 2-functor, then a 2-natural transformation ζ from F to G is a function sending

[1] Thanks to Hnaef on Wikimedia for the LaTeX-source file!

$\mathtt{i} \in \mathscr{A}$ to a 1-morphism $\zeta_{\mathtt{i}} \in \mathscr{C}$ such that for every 2-morphism $\alpha : f \to g$, where $f, g \in \mathscr{A}(\mathtt{i}, \mathtt{j})$, we have



In particular, applied to the identity 2-morphisms we get that ζ is an ordinary natural transformation between the associated ordinary functors F and G. Note that 2-categories with 2-functors and 2-natural transformations form a 2-category.

A 2-category is called *additive* if it is enriched over the category of additive categories. If $\Bbbk$ is a fixed field, a 2-category is called $\Bbbk$-*linear* if it is enriched over the category of $\Bbbk$-linear categories.

Now some notation: for a 2-category $\mathscr{C}$, objects of $\mathscr{C}$ will be denoted by $\mathtt{i}, \mathtt{j}$ and so on, objects of $\mathscr{C}(\mathtt{i}, \mathtt{j})$ (that is 1-morphisms) will be called f, g and so on, and 2-morphisms from f to g will be written α, β and so on. The identity 1-morphism in $\mathscr{C}(\mathtt{i}, \mathtt{i})$ will be denoted $\mathbb{1}_{\mathtt{i}}$ and the identity 2-morphism from f to f will be denoted id_f. Composition of 1-morphisms will be denoted by $\circ$, horizontal composition of 2-morphisms will be denoted by $\circ_0$ and vertical composition of 2-morphisms will be denoted by $\circ_1$.

2.5. **(Genuine) categorification.** Let $\mathscr{C}$ be an additive 2-category.

Definition 2.11. The *Grothendieck category* $[\mathscr{C}]$ of $\mathscr{C}$ is the category defined as follows: $[\mathscr{C}]$ has the same objects as $\mathscr{C}$, for $\mathtt{i}, \mathtt{j} \in [\mathscr{C}]$ we have $[\mathscr{C}](\mathtt{i}, \mathtt{j}) = [\mathscr{C}(\mathtt{i}, \mathtt{j})]$ and the multiplication of morphisms in $[\mathscr{C}]$ is given by $[M] \circ [N] := [M \circ_0 N]$.

Note that $[\mathscr{C}]$ is a preadditive category (i.e. it is enriched over the category of abelian groups). As before, let $\mathbb{F}$ be a commutative ring with 1.

Definition 2.12. The $\mathbb{F}$-*decategorification* $[\mathscr{C}]^{\mathbb{F}}$ of $\mathscr{C}$ is the category $\mathbb{F} \otimes_{\mathbb{Z}} [\mathscr{C}]$.

The category $[\mathscr{C}]^{\mathbb{F}}$ is $\mathbb{F}$-linear (i.e. enriched over $\mathbb{F}$-Mod) by definition. Now we are ready to define our central notion of (genuine) categorification.

Definition 2.13. Let $\mathcal{A}$ be an $\mathbb{F}$-linear category with at most countably many objects. A *categorification* of $\mathcal{A}$ is a pair $(\mathscr{A}, \varphi)$, where $\mathscr{A}$ is an additive 2-category and $\varphi : \mathcal{A} \to [\mathscr{A}]^{\mathbb{F}}$ is an isomorphism.

In the special case when $\mathcal{A}$ has only one object, say $\mathtt{i}$, the morphism set $\mathcal{A}(\mathtt{i}, \mathtt{i})$ is an $\mathbb{F}$-algebra. Therefore Definition 2.13 contains, as a special case, the definition of categorification for arbitrary $\mathbb{F}$-algebras.

Example 2.14. Let $A = \mathbb{C}[a]/(a^2 - 2a)$. For the algebra D of dual numbers consider the bimodule $X = D \otimes_{\mathbb{C}} D$ and denote by $\mathscr{C}$ the 2-category with one object $\mathtt{i} = D$-mod such that $\mathscr{C}(\mathtt{i}, \mathtt{i})$ is the full additive subcategory of the category of endofunctors of $\mathtt{i}$, consisting of all functors isomorphic to direct sums of copies of $\mathrm{Id} = \mathrm{Id}_{D\text{-mod}}$ and $\mathrm{F} = X \otimes_D {}_-$. It is easy to check that $X \otimes_D X \cong X \oplus X$, which implies that $\mathscr{C}(\mathtt{i}, \mathtt{i})$ is closed under composition of functors. The classes $[\mathrm{Id}]$ and $[\mathrm{F}]$ form a basis of $[\mathscr{C}(\mathtt{i}, \mathtt{i})]$. Since $\mathrm{F} \circ \mathrm{F} = \mathrm{F} \oplus \mathrm{F}$, it follows that the map $\varphi : A \to [\mathscr{C}(\mathtt{i}, \mathtt{i})]^{\mathbb{C}}$ such that $1 \mapsto [\mathrm{Id}]$ and $a \mapsto [\mathrm{F}]$ is an isomorphism. Hence $(\mathscr{C}, \varphi)$ is a $\mathbb{C}$-categorification of A.

The above leads to the following major problem:

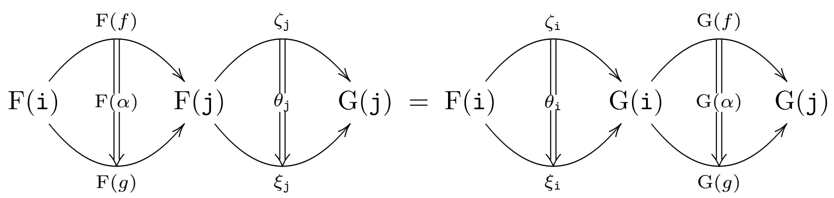
Problem 2.15. Given an $\mathbb{F}$-linear category $\mathcal{A}$ (satisfying some reasonable integrality conditions), construct a categorification of $\mathcal{A}$.

Among various solutions to this problem in some special cases one could mention [Ro1, Ro2, La, KhLa]. Whereas [Ro1, Ro2] propagates algebraic approach (using generators and relations), the approach of [La, KhLa] uses diagrammatic calculus and is motivated and influenced by topological methods. The idea to use 2-categories for a proper definition of algebraic categorification seems to go back at least to [Ro1] and is based on the results of [CR] which will be mentioned later on. One of the main advantages of this approach when compared with weak categorification is that now all extra properties (e.g. relations or involutions) for the generators of our algebra can be encoded into the internal structure of the 2-category. Thus relations between generators now can be interpreted as invertability of some 2-morphisms and biadjointness of elements connected by an anti-involution can be interpreted in terms of existence of adjunction morphisms, etc.

3. Basics: 2-representations of finitary 2-categories

3.1. **2-representations of 2-categories.** Let $\mathscr{C}$ be a 2-category. As usual, a 2-representation of a $\mathscr{C}$ is a 2-functor to some other 2-category. We will deal with $\Bbbk$-linear representations. Denote by $\mathfrak{R}_\Bbbk$ and $\mathfrak{D}_\Bbbk$ the 2-categories whose objects are categories equivalent to module categories of finite-dimensional $\Bbbk$-algebras and their (bounded) derived categories, respectively, 1-morphisms are functors between objects, and 2-morphisms are natural transformations of functors. Define the 2-categories $\mathscr{C}$-$\mathfrak{mod}$ and $\mathscr{C}$-$\mathfrak{dmod}$ of $\Bbbk$-*linear* 2-*representations* and $\Bbbk$-*linear triangulated* 2-*representations* of $\mathscr{C}$ as follows:

- Objects of $\mathscr{C}$-$\mathfrak{mod}$ (resp. $\mathscr{C}$-$\mathfrak{dmod}$) are 2-functors from $\mathscr{C}$ to $\mathfrak{R}_\Bbbk$ (resp. $\mathfrak{D}_\Bbbk$);
- 1-morphisms are 2-natural transformations (these are given by a collection of functors);
- 2-morphisms are the so-called *modifications*, defined as follows: Let $\mathbf{M}, \mathbf{N} \in \mathscr{C}$-$\mathfrak{mod}$ and $\zeta, \xi : \mathbf{M} \to \mathbf{N}$ be 2-natural transformations. A modification $\theta : \zeta \to \xi$ is a function, which assigns to every $\mathtt{i} \in \mathscr{C}$ a 2-morphism $\theta_{\mathtt{i}} : \zeta_{\mathtt{i}} \to \xi_{\mathtt{i}}$ such that for every 1-morphisms $f, g \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and any 2-morphism $\alpha : f \to g$ we have



For simplicity we will identify objects in $\mathscr{C}(\mathtt{i}, \mathtt{k})$ with their images under a 2-representation (i.e. we will use the module notation). We will also use 2-*action* and 2-*module* as a synonym for 2-*representation*. Now we can define genuine categorifications for A-modules.

Definition 3.1. Let $\Bbbk$ be a field, A a $\Bbbk$-linear category with at most countably many objects and M an A-module. A *(pre)categorification* of M is a tuple $(\mathscr{A}, \mathbf{M}, \varphi, \psi)$, where

- $(\mathscr{A}, \varphi)$ is a categorification of A;
- $\mathbf{M} \in \mathscr{A}$-$\mathfrak{mod}$ or $\mathbf{M} \in \mathscr{A}$-$\mathfrak{dmod}$ is such that for every $\mathtt{i}, \mathtt{j} \in \mathscr{A}$ and $a \in \mathscr{A}(\mathtt{i}, \mathtt{j})$ the functor $\mathbf{M}(a)$ is exact or triangulated, respectively;

- $\psi = (\psi_{\mathtt{i}})_{\mathtt{i}\in\mathscr{A}}$, where every $\psi_{\mathtt{i}} : M(\mathtt{i}) \to [\mathbf{M}(\mathtt{i})]$ is a monomorphism such that for every $\mathtt{i},\mathtt{j} \in \mathscr{A}$ and $a \in \mathscr{A}(\mathtt{i},\mathtt{j})$ the following diagram commutes:

$$\begin{array}{ccc} M(\mathtt{i}) & \xrightarrow{M(\varphi^{-1}([a]))} & M(\mathtt{j}) \\ {\scriptstyle\psi_{\mathtt{i}}}\downarrow & & \downarrow{\scriptstyle\psi_{\mathtt{j}}} \\ [\mathbf{M}(\mathtt{i})]^{\Bbbk} & \xrightarrow{[\mathbf{M}(a)]} & [\mathbf{M}(\mathtt{j})]^{\Bbbk}. \end{array}$$

If every $\psi_{\mathtt{i}}$ is an isomorphism, $(\mathscr{A},\mathbf{M},\varphi,\psi)$ is called a *categorification* of M.

An important feature of this definition is that categorification of all relations and other structural properties of A is encoded into the internal structure of the 2-category $\mathscr{A}$. In particular, the requirement for $\mathbf{M}(a)$ to be structural is often automatically satisfied because of the existence of adjunctions in $\mathscr{A}$ (see the next subsection). Later on we will see many examples of categorifications of modules over various algebras. For the moment we would like to look at the other direction: to be able to categorify modules one should develop some abstract 2-representation theory of 2-categories. Some overview of this (based on [MM2]) is the aim of this section.

3.2. **Fiat-categories.** Let $\Bbbk$ be a field. An additive 2-category $\mathscr{C}$ with involution $*$ is called a *fiat-category* (over $\Bbbk$) provided that

(I) $\mathscr{C}$ has finitely many objects;
(II) for every $\mathtt{i},\mathtt{j} \in \mathscr{C}$ the category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is fully additive with finitely many isomorphism classes of indecomposable objects;
(III) for every $\mathtt{i},\mathtt{j} \in \mathscr{C}$ the category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is enriched over $\Bbbk$-mod;
(IV) for every $\mathtt{i} \in \mathscr{C}$ the identity object in $\mathscr{C}(\mathtt{i},\mathtt{i})$ is indecomposable;
(V) for any $\mathtt{i},\mathtt{j} \in \mathscr{C}$ and any 1-morphism $f \in \mathscr{C}(\mathtt{i},\mathtt{j})$ there exist 2-morphisms $\alpha : f \circ f^* \to \mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}} \to f^* \circ f$ such that $(\alpha \circ_0 \mathrm{id}_f) \circ_1 (\mathrm{id}_f \circ_0 \beta) = \mathrm{id}_f$ and $(\mathrm{id}_{f^*} \circ_0 \alpha) \circ_1 (\beta \circ_0 \mathrm{id}_{f^*}) = \mathrm{id}_{f^*}$.

If $\mathscr{C}$ is a fiat category and $\mathbf{M}$ is a 2-representation of $\mathscr{C}$, then for every 1-morphism f the functor $\mathbf{M}(f)$ is both left and right adjoint to $\mathbf{M}(f^*)$, in particular, both are exact if $\mathbf{M} \in \mathscr{C}\text{-}\mathfrak{mod}$.

Example 3.2. The category $\mathscr{C}$ from Example 2.14 is a fiat-category.

3.3. **Principal 2-representations of fiat-categories.** Let $\mathscr{C}$ be a finitary 2-category. For $\mathtt{i},\mathtt{j} \in \mathscr{C}$ denote by $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ the category defined as follows: Objects of $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ are diagrams of the form $f \xrightarrow{\alpha} g$, where $f,g \in \mathscr{C}(\mathtt{i},\mathtt{j})$ are 1-morphisms and α is a 2-morphism. Morphisms of $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ are equivalence classes of diagrams as given by the solid part of the following picture:

$$\begin{array}{ccc} f & \xrightarrow{\alpha} & g \\ {\scriptstyle\beta}\downarrow & \overset{\xi}{\swarrow} & \downarrow{\scriptstyle\beta'} \\ f' & \xrightarrow{\alpha'} & g' \end{array} \quad , \qquad f,f',g,g' \in \mathscr{C}(\mathtt{i},\mathtt{j}),$$

modulo the ideal generated by all morphisms for which there exists ξ as shown by the dotted arrow above such that $\alpha'\xi = \beta'$. As $\mathscr{C}$ is a finitary category, the category $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ is abelian and equivalent to $\mathcal{C}_{\mathtt{i},\mathtt{j}}^{\mathrm{op}}$-mod, see [Fre].

For $\mathtt{i} \in \mathscr{C}$ define the 2-functor $\mathbf{P}_{\mathtt{i}} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ as follows: for $\mathtt{j} \in \mathscr{C}$ set $\mathbf{P}_{\mathtt{i}}(\mathtt{j}) = \overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$. Further, for $\mathtt{k} \in \mathscr{C}$ and $f \in \mathscr{C}(\mathtt{j},\mathtt{k})$ the left horizontal composition with (the identity on) f defines a functor from $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ to $\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})$. We define this functor to be $\mathbf{P}_{\mathtt{i}}(f)$. Given a 2-morphism $\alpha : f \to g$, the left horizontal composition with

α gives a natural transformation from $\mathbf{P}_{\mathtt{i}}(f)$ to $\mathbf{P}_{\mathtt{i}}(g)$. We define this natural transformation to be $\mathbf{P}_{\mathtt{i}}(\alpha)$. From the definition it follows that $\mathbf{P}_{\mathtt{i}}$ is a strict 2-functor from $\mathscr{C}$ to $\mathfrak{R}_{\Bbbk}$. The 2-representation $\mathbf{P}_{\mathtt{i}}$ is called the $\mathtt{i}$-*th principal* 2-representation of $\mathscr{C}$.

For $\mathtt{i},\mathtt{j}\in\mathscr{C}$ and a 1-morphism $f\in\mathscr{C}(\mathtt{i},\mathtt{j})$ we denote by P_f the projective object $0\to f$ of $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$.

Proposition 3.3 (The universal property of $\mathbf{P}_{\mathtt{i}}$.)**.** *Let* $\mathbf{M}$ *be a* 2*-representation of* $\mathscr{C}$ *and* $M\in\mathbf{M}(\mathtt{i})$.

(a) *For* $\mathtt{j}\in\mathscr{C}$ *define the functor* $\Phi^M_{\mathtt{j}}:\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})\to\mathbf{M}(\mathtt{j})$ *as follows:* $\Phi^M_{\mathtt{j}}$ *sends a diagram* $f\xrightarrow{\alpha}g$ *in* $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ *to the cokernel of* $\mathbf{M}(\alpha)$*. Then* $\Phi^M=(\Phi^M_{\mathtt{j}})_{\mathtt{j}\in\mathscr{C}}$ *is the unique morphism from* $\mathbf{P}_{\mathtt{i}}$ *to* $\mathbf{M}$ *sending* $P_{\mathbb{1}_{\mathtt{i}}}$ *to* M.

(b) *The correspondence* $M\mapsto\Phi^M$ *is functorial.*

Idea of the proof. This follows from the 2-functoriality of $\mathbf{M}$. □

3.4. **Cells.** Let $\mathscr{C}$ be a fiat-category. Set $\mathcal{C}=\cup_{\mathtt{i},\mathtt{j}}\mathcal{C}_{\mathtt{i},\mathtt{j}}$. Let $\mathtt{i},\mathtt{j},\mathtt{k},\mathtt{l}\in\mathscr{C}$, $f\in\mathcal{C}_{\mathtt{i},\mathtt{j}}$ and $g\in\mathcal{C}_{\mathtt{k},\mathtt{l}}$. We will write $f\leq_R g$ provided that there exists $h\in\mathscr{C}(\mathtt{j},\mathtt{l})$ such that g occurs as a direct summand of $h\circ f$ (note that this is possible only if $\mathtt{i}=\mathtt{k}$). Similarly, we will write $f\leq_L g$ provided that there exists $h\in\mathscr{C}(\mathtt{k},\mathtt{i})$ such that g occurs as a direct summand of $f\circ h$ (note that this is possible only if $\mathtt{j}=\mathtt{l}$). Finally, we will write $f\leq_{LR} g$ provided that there exists $h_1\in\mathscr{C}(\mathtt{k},\mathtt{i})$ and $h_2\in\mathscr{C}(\mathtt{j},\mathtt{l})$ such that g occurs as a direct summand of $h_2\circ f\circ h_1$. The relations $\leq_L$, $\leq_R$ and $\leq_{LR}$ are partial preorders on $\mathcal{C}$. The map $f\mapsto f^*$ preserves $\leq_{LR}$ and swaps $\leq_L$ and $\leq_R$.

For $f\in\mathcal{C}$ the set of all $g\in\mathcal{C}$ such that $f\leq_R g$ and $g\leq_R f$ will be called the *right cell* of f and denoted by $\mathcal{R}_f$. The *left cell* $\mathcal{L}_f$ and the *two-sided cell* $\mathcal{LR}_f$ are defined analogously.

Example 3.4. The 2-category $\mathscr{C}$ from Example 2.14 has two right cells, namely $\{\mathrm{Id}\}$ and $\{\mathrm{F}\}$, which are also left cells and thus two-sided cells as well.

3.5. **Cells modules.** Let $\mathscr{C}$ be a fiat-category. For $\mathtt{i},\mathtt{j}\in\mathscr{C}$ indecomposable projective modules in $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ are indexed by objects of $\mathcal{C}_{\mathtt{i},\mathtt{j}}$. For $f\in\mathcal{C}_{\mathtt{i},\mathtt{j}}$ we denote by L_f the unique simple quotient of P_f.

Proposition 3.5. (a) *For* $f,g\in\mathcal{C}$ *the inequality* $f\,L_g\neq 0$ *is equivalent to* $f^*\leq_L g$.

(b) *For* $f,g,h\in\mathcal{C}$ *the inequality* $[f\,L_g:L_h]\neq 0$ *implies* $h\leq_R g$.

(c) *For* $g,h\in\mathcal{C}$ *such that* $h\leq_R g$ *there exists* $f\in\mathcal{C}$ *such that* $[f\,L_g:L_h]\neq 0$.

(d) *Let* $f,g,h\in\mathcal{C}$*. If* L_f *occurs in the top or in the socle of* $h\,L_g$*, then* $f\in\mathcal{R}_g$.

(e) *For any* $f\in\mathcal{C}_{\mathtt{i},\mathtt{j}}$ *there is a unique (up to scalar) nontrivial homomorphism from* $P_{\mathbb{1}_{\mathtt{i}}}$ *to* f^*L_f*. In particular,* $f^*L_f\neq 0$.

Idea of the proof. To prove (a), without loss of generality we may assume $g\in\mathcal{C}_{\mathtt{i},\mathtt{j}}$ and $f\in\mathcal{C}_{\mathtt{j},\mathtt{k}}$. Then $f\,L_g\neq 0$ if and only if there is $h\in\mathcal{C}_{\mathtt{i},\mathtt{k}}$ such that $\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(P_h,f\,L_g)\neq 0$. Using $P_h=h\,P_{\mathbb{1}_{\mathtt{i}}}$ and adjunction we obtain

$$0\neq\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(P_h,f\,L_g)=\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(f^*\circ h\,P_{\mathbb{1}_{\mathtt{i}}},L_g).$$

This inequality is equivalent to the claim that $P_g=g\,P_{\mathbb{1}_{\mathtt{i}}}$ is a direct summand of $f^*\circ h\,P_{\mathbb{1}_{\mathtt{i}}}$, that is g is a direct summand of $f^*\circ h$. Claim (a) follows. Other claims are proved similarly □

Fix $\mathtt{i}\in\mathscr{C}$. Let $\mathcal{R}$ be a right cell in $\mathcal{C}$ such that $\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}\neq\varnothing$ for some $\mathtt{j}\in\mathscr{C}$.

Proposition 3.6. (a) *There is a unique submodule* $K=K_{\mathcal{R}}$ *of* $P_{\mathbb{1}_{\mathtt{i}}}$ *which has the following properties:*

(i) *Every simple subquotient of* $P_{\mathbb{1}_{\mathtt{i}}}/K$ *is annihilated by any* $f\in\mathcal{R}$.

(*ii*) *The module K has simple top, which we denote by $L_{g_{\mathcal{R}}}$, and $f\,L_{g_{\mathcal{R}}} \neq 0$ for any $f \in \mathcal{R}$.*

(*b*) *For any $f \in \mathcal{R}$ the module $f\,L_{g_{\mathcal{R}}}$ has simple top L_f.*

(*c*) *We have $g_{\mathcal{R}} \in \mathcal{R}$.*

(*d*) *For any $f \in \mathcal{R}$ we have $f^* \leq_L g_{\mathcal{R}}$ and $f \leq_R g_{\mathcal{R}}^*$.*

(*e*) *We have $g_{\mathcal{R}}^* \in \mathcal{R}$.*

Idea of the proof. The module K is defined as the minimal module with property (ai). Then for every $f \in \mathcal{R}$ we have $f\,K = f\,P_{\mathbb{1}_{\mathtt{i}}} = P_f$. The latter module has simple top, which implies that K has simple top. The rest follows from Proposition 3.5. □

Set $L = L_{\mathrm{G}_{\mathcal{R}}}$. For $\mathtt{j} \in \mathscr{C}$ denote by $\mathcal{D}_{\mathcal{R},\mathtt{j}}$ the full subcategory of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ with objects $\mathrm{G}\,L$, $\mathrm{G} \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}$. As each $\mathrm{G}\,L$ is a quotient of P_{G} and 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between projective modules in $\mathbf{P}_{\mathtt{i}}$ (see Subsection 2.3), it follows that 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between the various $\mathrm{G}\,L$.

Theorem 3.7 (Construction of cell modules.)**.** (*a*) *For every $f \in \mathcal{C}$ and $g \in \mathcal{R}$, the module $f \circ g\,L$ is isomorphic to a direct sum of modules of the form $h\,L$, $h \in \mathcal{R}$.*

(*b*) *For every $f, h \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}$ we have*

$$\dim \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(f\,L, h\,L) = [h\,L : L_f].$$

(*c*) *For $f \in \mathcal{R}$ let Ker_f be the kernel of $P_f \twoheadrightarrow f\,L$. Then the module $\oplus_{f\in\mathcal{R}}\mathrm{Ker}_f$ is stable under any endomorphism of $\oplus_{f\in\mathcal{R}}P_f$.*

(*d*) *The full subcategory $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ consisting of all modules M which admit a two step resolution $X_1 \to X_0 \twoheadrightarrow M$, $X_1, X_0 \in \mathrm{add}(\oplus_{f\in\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}} f\,L)$, is equivalent to $\mathcal{D}_{\mathcal{R},\mathtt{j}}^{\mathrm{op}}$-mod.*

(*e*) *Restriction from $\mathbf{P}_{\mathtt{i}}$ defines the structure of a 2-representation of $\mathscr{C}$ on $\mathbf{C}_{\mathcal{R}}$, which is called the* cell module *corresponding to $\mathcal{R}$.*

Example 3.8. Consider the category $\mathscr{C}$ from Example 2.14. For the cell representation $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}$ we have $G_{\{\mathbb{1}_{\mathtt{i}}\}} = \mathbb{1}_{\mathtt{i}}$, which implies that $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}(\mathtt{i}) = \mathbb{C}$-mod; $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}(\mathrm{F}) = 0$ and $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}(\alpha) = 0$ for all radical 2-morphisms α. For the cell representation $\mathbf{C}_{\{\mathrm{F}\}}$ we have $G_{\{\mathrm{F}\}} = \mathrm{F}$, which implies that $\mathbf{C}_{\{\mathrm{F}\}}(\mathtt{i}) = D$-mod, $\mathbf{C}_{\{\mathrm{F}\}}(\mathrm{F}) = \mathrm{F}$ and $\mathbf{C}_{\{\mathrm{F}\}}(\alpha) = \alpha$ for all radical 2-morphisms α.

3.6. **Homomorphisms from a cell module.** Let $\mathscr{C}$ be a fiat category, $\mathcal{R}$ a right cell in $\mathscr{C}$ and $\mathtt{i} \in \mathscr{C}$ be such that $g_{\mathcal{R}} \in \mathcal{C}_{\mathtt{i},\mathtt{i}}$. Let further $f \in \mathscr{C}(\mathtt{i},\mathtt{i})$ and $\alpha : f \to g_{\mathcal{R}}$ be such that $\mathbf{P}_{\mathtt{i}}(\alpha) : f\,P_{\mathbb{1}_{\mathtt{i}}} \to g_{\mathcal{R}}\,P_{\mathbb{1}_{\mathtt{i}}}$ gives a projective presentation of $L_{g_{\mathcal{R}}}$.

Theorem 3.9. *Let $\mathbf{M}$ be a 2-representation of $\mathscr{C}$. Denote by $\Theta = \Theta_{\mathcal{R}}^{\mathbf{M}}$ the cokernel of $\mathbf{M}(\alpha)$.*

(*a*) *The functor Θ is a right exact endofunctor of $\mathbf{M}(\mathtt{i})$.*

(*b*) *For every morphism Ψ from $\mathbf{C}_{\mathcal{R}}$ to $\mathbf{M}$ we have $\Psi(L_{g_{\mathcal{R}}}) \in \Theta(\mathbf{M}(\mathtt{i}))$.*

(*c*) *For every $M \in \Theta(\mathbf{M}(\mathtt{i}))$ there is a unique morphism Ψ^M from $\mathbf{C}_{\mathcal{R}}$ to $\mathbf{M}$ sending $L_{g_{\mathcal{R}}}$ to M.*

(*d*) *The correspondence $M \mapsto \Psi^M$ is functorial in $M \in \Theta(\mathbf{M}(\mathtt{i}))$.*

Idea of the proof. Using the universal property of $\mathbf{P}_{\mathtt{i}}$, this follows from the 2-functoriality of $\mathbf{M}$. □

3.7. **Serre subcategories and quotients.** Let $\mathscr{C}$ be a fiat category and $\mathbf{M} \in \mathscr{C}$-$\mathfrak{mod}$. In every $\mathbf{M}(\mathtt{i})$ choose a set of simple modules and denote by $\mathbf{N}(\mathtt{i})$ the Serre subcategory of $\mathbf{M}(\mathtt{i})$ which these modules generate. If $\mathbf{N}$ turns out to be stable with respect to the action of $\mathscr{C}$ (restricted from $\mathbf{M}$), then $\mathbf{N}$ become a 2-representation of $\mathscr{C}$ (a *Serre submodule*). Moreover, the quotient $\mathbf{Q}$, defined via $\mathbf{Q}(\mathtt{i}) := \mathbf{M}(\mathtt{i})/\mathbf{N}(\mathtt{i})$, also carries the natural structure of a 2-representation of $\mathscr{C}$.

Example 3.10. The cell module $\mathbf{C}_{\{\mathbb{1}_\mathtt{i}\}}$ in Example 3.8 is a Serre submodule of $\mathbf{P}_\mathtt{i}$ and the cell module $\mathbf{C}_{\{\mathrm{F}\}}$ is the corresponding quotient.

Serre submodules of $\mathbf{M}$ can be organized into a *Serre filtration* of $\mathbf{M}$.

3.8. **Naturally commuting functors.** A 2-morphism between two 2-representations of some 2-category $\mathscr{C}$ can be understood via functors *naturally commuting* with the functors defining the action of $\mathscr{C}$ in the terminology of [Kh]. Note that this notion is not symmetric, that is if some functor F naturally commutes with the action of $\mathscr{C}$ it does not follow that an element of this action naturally commutes with F (in fact, the latter does not really make sense as F is not specified as an object of some 2-action of a 2-category).

Another interesting notion from [Kh] is that of a category with full projective functors, which just means that we have a 2-representation of a 2-category $\mathscr{C}$ such that 2-morphisms of $\mathscr{C}$ surject onto homomorphisms between projective modules of this representation.

4. Category $\mathcal{O}$: definitions

4.1. **Definition of category $\mathcal{O}$.** One of the main sources for categorification models is the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ from [BGG2]. Let us give a short recap of its properties (see also [Di, Hu]).

For $n \in \mathbb{N}$ consider the reductive complex Lie algebra $\mathfrak{g} = \mathfrak{g}_n = \mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. Let $\mathfrak{h}$ denote the commutative subalgebra of all diagonal matrices (the *Cartan subalgebra* of $\mathfrak{g}$). Denote by $\mathfrak{n}_+$ and $\mathfrak{n}_-$ the Lie subalgebras of upper and lower triangular matrices, respectively. Then we have the *standard triangular decomposition* $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is the *Borel subalgebra* of $\mathfrak{g}$. For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

Definition 4.1. The category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ is the full subcategory of the category of $\mathfrak{g}$-modules which consists of all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $\mathfrak{h}$ on M is diagonalizable;
- the action of $U(\mathfrak{n}_+)$ on M is locally finite, that is $\dim U(\mathfrak{n}_+)v < \infty$ for all $v \in M$.

For example, all semi-simple finite-dimensional $\mathfrak{g}$-modules are objects of $\mathcal{O}$. Elements of $\mathfrak{h}^*$ are called *weights*. For a $\mathfrak{g}$-module M and $\lambda \in \mathfrak{h}^*$ define the corresponding weight space

$$M_\lambda := \{v \in M : hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Then the condition of $\mathfrak{h}$-diagonalizability can be written in the following form:

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

For $i, j \in \{1, 2, \dots, n\}$ denote by e_{ij} the corresponding matrix unit. Then $\{e_{ii}\}$ form a standard basis of $\mathfrak{h}$. For $i < j$ set $\alpha_{ij} = e_{ii}^* - e_{jj}^*$. Then $\mathbf{R} := \{\pm\alpha_{ij}\}$ is a root system of type A_n (in its linear hull). Let $W \cong \mathbb{S}_n$ be the corresponding Weyl group. It acts on $\mathfrak{h}^*$ (and $\mathfrak{h}$) by permuting indexes of elements of the standard basis. For a root α we denote by s_α the corresponding reflection in W. For $i = 1, 2, \dots, n-1$ we also denote by s_i the simple reflection $s_{\alpha_{ii+1}}$. We denote by S the set of all simple reflections. The corresponding positive roots form a basis of $\mathbf{R}$. Let ρ be the half of the sum of all positive roots. Define the *dot-action* of W on $\mathfrak{h}^*$ as follows: $w\cdot\lambda = w(\lambda+\rho)-\rho$.

For any M, λ and $i<j$ we have $e_{ij}M_\lambda \subset M_{\lambda+\alpha_{ij}}$ and $e_{ji}M_\lambda \subset M_{\lambda-\alpha_{ij}}$. For $i<j$ we have positive roots $\{\alpha_{ij}\}$ and negative roots $\{-\alpha_{ij}\}$, moreover, the matrix units e_{ij} and e_{ji} are root element for roots α_{ij} and $-\alpha_{ij}$, respectively. In particular, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the linear spans of all positive and negative root spaces, respectively. With respect to the system S we have the length function $\mathfrak{l}: W\to\{0,1,2,\dots\}$. We denote by w_o the longest element of W and by $\leq$ the Bruhat order on W.

Define the *standard* partial order $\leq$ on $\mathfrak{h}^*$ as follows: $\lambda\leq\mu$ if and only if $\mu-\lambda$ is a linear combination of positive roots with non-negative integral coefficients. Let $\mathfrak{h}^*_{\mathrm{dom}}$ denote the set of all elements in $\mathfrak{h}^*$ dominant with respect to the dot-action.

4.2. **Verma modules.** For $\lambda\in\mathfrak{h}^*$ let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{h}$-module on which elements of $\mathfrak{h}$ act via λ. Setting $\mathfrak{n}_+\mathbb{C}_\lambda=0$ defines on $\mathbb{C}_\lambda$ the structure of a $\mathfrak{b}$-module. The induced module

$$M(\lambda) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

is called the *Verma module* with highest weight λ, see [Ve, BGG1, Hu] and [Di, Chapter 7]. By adjunction, $M(\lambda)$ has the following universal property: for any $\mathfrak{g}$-module N we have

$$\begin{array}{rcc} \mathrm{Hom}_{\mathfrak{g}}(M(\lambda),N) & \cong & \{v\in N_\lambda : \mathfrak{n}_+v=0\} \\ \varphi & \mapsto & \varphi(1\otimes 1). \end{array}$$

A weight vector v satisfying $\mathfrak{n}_+v=0$ is called a *highest weight* vector. A module generated by a highest weight vector is called a *highest weight* module.

It is easy to check that $M(\lambda)\in\mathcal{O}$ and that $M(\lambda)_\lambda=\mathbb{C}\langle 1\otimes 1\rangle$. Further, $M(\lambda)$ has the unique maximal submodule, namely the sum of all submodules which do not intersect $M(\lambda)_\lambda$. Hence $M(\lambda)$ has the unique simple quotient denoted by $L(\lambda)$. Obviously $L(\lambda)\in\mathcal{O}$. The set $\{L(\lambda):\lambda\in\mathfrak{h}^*\}$ is a complete and irredundant set of simple highest weight modules (which are exactly the simple objects in $\mathcal{O}$).

Theorem 4.2. (*a*) *$M(\lambda)$ has finite length.*
(*b*) $[M(\lambda):L(\lambda)]=1$ *and* $[M(\lambda):L(\mu)]\neq 0$ *implies* $\mu\leq\lambda$.
(*c*) *$M(\lambda)$ has simple socle.*
(*d*) $\dim\mathrm{Hom}_{\mathfrak{g}}(M(\lambda):M(\mu))\leq 1$ *and any nonzero element of this space is injective.*
(*e*) *The following conditions are equivalent:*
  (*i*) $\dim\mathrm{Hom}_{\mathfrak{g}}(M(\mu):M(\lambda))\neq 0$.
  (*ii*) $[M(\lambda):L(\mu)]\neq 0$.
  (*iii*) *There is a sequence $\beta_1,\dots,\beta_k$ of positive roots such that*

$$\lambda\geq s_{\beta_1}\cdot\lambda\geq s_{\beta_2}s_{\beta_1}\cdot\lambda\geq\dots\geq s_{\beta_k}\cdots s_{\beta_2}s_{\beta_1}\cdot\lambda=\mu.$$

(*f*) *Every endomorphism of $M(\lambda)$ is scalar.*

4.3. **Block decomposition.** Denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. By Theorem 4.2(f), the action of $Z(\mathfrak{g})$ on $M(\lambda)$ is scalar. Let $\chi_\lambda: Z(\mathfrak{g})\to\mathbb{C}$ be the corresponding *central character*. The map $\lambda\mapsto\chi_\lambda$ sets up a bijection between $\mathfrak{h}^*_{\mathrm{dom}}$ and the set of central characters (see [Di, Section 7.4]). For a dominant weight λ denote by $\mathcal{O}_\lambda$ the full subcategory of $\mathcal{O}$ consisting of all modules on which the kernel of χ_λ acts locally nilpotently.

Theorem 4.3. *Every simple module in $\mathcal{O}_\lambda$ has the form $L(w\cdot\lambda)$ for some $w\in W$ and we have*

$$\mathcal{O} = \bigoplus_{\lambda\in\mathfrak{h}^*_{\mathrm{dom}}} \mathcal{O}_\lambda.$$

In particular, it follows that every object in $\mathcal{O}$ has finite length. Categories $\mathcal{O}_\lambda$ are called *blocks* of $\mathcal{O}$. For $\lambda = 0$ the block $\mathcal{O}_0$ contains the (unique up to isomorphism) one-dimensional $\mathfrak{g}$-module and is called the *principal* block of $\mathcal{O}$. Note that blocks $\mathcal{O}_\lambda$ might be further decomposable. To remedy this, in what follows we will restrict our attention to *integral* blocks, that is blocks corresponding to λ, the coordinates of which in the standard basis have integral differences. By [So1], any block is equivalent to an integral block, possibly for a smaller n. Note that, for integral $\mu \in \mathfrak{h}^*$, the Verma module $M(\mu)$ is simple if and only if μ is dot-antidominant, further, $M(\mu)$ is projective if and only if μ is dot-dominant.

An integral block $\mathcal{O}_\lambda$ is called *regular* if the orbit $W \cdot \lambda$ is regular. In this case simple objects in the block are bijectively indexed by elements of W. An integral block $\mathcal{O}_\lambda$ is called *singular* if it is not regular. Let W_λ denote the dot-stabilizer of λ in W. Then simple objects in the singular block $\mathcal{O}_\lambda$ are bijectively indexed by cosets W/W_λ, or , alternatively, by longest coset representatives in these cosets.

Theorem 4.4. *Every $\mathcal{O}_\lambda$ has enough projectives, in particular, $\mathcal{O}_\lambda$ is equivalent to the category B_λ-*mod *of finite dimensional modules over some finite dimensional associative algebra B_λ.*

For $\mu \in \mathfrak{h}^*$ we denote by $P(\mu)$ and $I(\mu)$ the indecomposable projective cover and injective hull of $L(\mu)$ in $\mathcal{O}$, respectively. For $\lambda \in \mathfrak{h}^*_{\text{dom}}$ set $P_\lambda := \oplus_{\mu \in W\cdot\lambda} P(\mu)$. Then P_λ is a projective generator of $\mathcal{O}_\lambda$ and hence we can take $B_\lambda = \mathrm{End}_{\mathfrak{g}}(P_\lambda)^{\mathrm{op}}$.

4.4. **BGG duality and quasi-hereditary structure.** A module $N \in \mathcal{O}$ is said to have a *standard filtration* or *Verma flag* if there is a filtration of N whose subquotients are Verma modules. The category of all modules in $\mathcal{O}$ which have a standard filtration is denoted by $\mathcal{F}(\Delta)$.

Theorem 4.5 (BGG reciprocity)**.** (*a*) *Every projective module in $\mathcal{O}$ has a standard filtration.*

(*b*) *If some $N \in \mathcal{O}$ has a standard filtration, then for any $\mu \in \mathfrak{h}^*$ the multiplicity $[N : M(\mu)]$ of $M(\mu)$ as a subquotient of a standard filtration of N does not depend on the choice of such filtration.*

(*c*) *For every $\mu, \nu \in \mathfrak{h}^*$ we have $[P(\mu) : M(\nu)] = [M(\nu) : L(\mu)]$.*

(*d*) *We have $[P(\mu) : M(\mu)] = 1$ and $[P(\mu) : M(\nu)] \neq 0$ implies $\nu \geq \mu$.*

The claims of Theorem 4.5(d) and Theorem 4.2(b) literally mean that for $\lambda \in \mathfrak{h}^*_{\text{dom}}$ the category $\mathcal{O}_\lambda$ is a *highest weight* category in the sense of [CPS1], and the algebra B_λ is *quasi-hereditary* in the sense of [DR] with Verma modules being *standard modules* for this quasi-hereditary structure. Traditionally, standard modules for quasi-hereditary algebras are denoted $\Delta(\mu) = M(\mu)$.

The category $\mathcal{O}$ has a *simple preserving duality* $\star$, that is an involutive contravariant equivalence which preserves isoclasses of simple modules. This implies that $B_\lambda \cong B_\lambda^{\mathrm{op}}$ such that the isomorphism preserves the equivalence classes of primitive idempotents. We have $L(\mu)^\star = L(\mu)$ and $P(\mu)^\star \cong I(\mu)$. The modules $\nabla(\mu) = \Delta(\mu)^\star$ are called *dual Verma* modules. Applying $\star$ to Theorem 4.5 we, in particular, have that every injective in $\mathcal{O}$ has a filtration whose subquotients are dual Verma modules (the so-called *costandard filtration*). The category of all modules in $\mathcal{O}$ which have a costandard filtration is denoted by $\mathcal{F}(\nabla)$.

Corollary 4.6. *For every $\lambda \in \mathfrak{h}^*_{\text{dom}}$ the block $\mathcal{O}_\lambda$ has finite global dimension.*

Idea of the proof. Using induction with respect to $\leq$ and Theorem 4.5(a), show that every Verma module has finite projective dimension. Then, using the opposite induction with respect to $\leq$ and Theorem 4.2(b) show that every simple module has finite projective dimension. □

One can show that the global dimension of $\mathcal{O}_0$ equals $2\mathfrak{l}(w_o)$, see e.g. [Ma2]. In fact, there are explicit formulae for the projective dimension of all structural modules in $\mathcal{O}_0$, see [Ma2, Ma3]. Another important homological feature of category $\mathcal{O}$, as a highest weight category, is:

Proposition 4.7. $\mathcal{F}(\Delta) = \{N \in \mathcal{O} : \mathrm{Ext}^i_{\mathcal{O}}(N, \nabla(\mu)) = 0$ *for all* $\mu \in \mathfrak{h}^*, i > 0\}$.

4.5. **Tilting modules and Ringel self-duality.** One of the nicest properties of quasi-hereditary algebras is existence of the so-called *tilting modules*, established in [Ri] (see also [CI] for the special case of the category $\mathcal{O}$).

Theorem 4.8. *For every* $\mu \in \mathfrak{h}^*$ *there is a unique (up to isomorphism) indecomposable module* $T(\mu) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ *such that* $\Delta(\mu) \subset T(\mu)$ *and the cokernel of this inclusion admits a standard filtration.*

For $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ the module $T_\lambda := \oplus_{\mu \in W\cdot\lambda} T(\mu)$ is called the *characteristic tilting module*. The word *tilting* is justified by the following property:

Theorem 4.9. *The module* T_λ *is ext-selforthogonal, has finite projective dimension and there is an exact sequence*

$$0 \to P_\lambda \to Q_0 \to Q_1 \to \cdots \to Q_k \to 0$$

such that $Q_i \in \mathrm{add}(T_\lambda)$ *for all* i.

The (opposite of the) endomorphism algebra of T_λ is called the *Ringel dual* of B_λ. The Ringel dual is defined for any quasi-hereditary algebra and is again a quasi-hereditary algebra. However, for category $\mathcal{O}$ we have the following *Ringel self-duality*:

Theorem 4.10 ([So3])**.** $\mathrm{End}_{\mathfrak{g}}(T_\lambda) = B_\lambda$.

For every $\mu \in \mathfrak{h}^*$ we have $T(\mu) = T(\mu)^*$ and hence tilting modules in $\mathcal{O}$ are also *cotilting*.

4.6. **Parabolic category $\mathcal{O}$.** Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra containing $\mathfrak{b}$. In what follows we only consider such parabolic subalgebras. Define $\mathcal{O}^{\mathfrak{p}}$ as the full subcategory of $\mathcal{O}$ consisting of all modules, on which the action of $U(\mathfrak{p})$ is locally finite. Alternatively, one could consider modules which decompose into a direct sum of finite dimensional modules, when restricted to the Levi factor of $\mathfrak{p}$. For example, taking $\mathfrak{g} = \mathfrak{p}$ the category $\mathcal{O}^{\mathfrak{p}}$ becomes the subcategory of finite dimensional modules in $\mathcal{O}$.

From the definition it follows that $\mathcal{O}^{\mathfrak{p}}$ is a Serre subcategory of $\mathcal{O}$ and thus is uniquely determined by simple modules which it contains. Let $\mathfrak{i}_{\mathfrak{p}} : \mathcal{O}^{\mathfrak{p}} \to \mathcal{O}$ be the natural inclusion, which is obviously exact. Further, denote by $\mathrm{Z}_{\mathfrak{p}} : \mathcal{O} \to \mathcal{O}^{\mathfrak{p}}$ and $\hat{\mathrm{Z}}_{\mathfrak{p}} : \mathcal{O} \to \mathcal{O}^{\mathfrak{p}}$ the left and the right adjoints to $\mathfrak{i}_{\mathfrak{p}}$, respectively. The functor $\mathrm{Z}_{\mathfrak{p}}$ is called the *Zuckerman* functor and $\hat{\mathrm{Z}}_{\mathfrak{p}}$ is called the *dual Zuckerman* functor (see [Zu]). These functors can be described as the functors of taking the maximal quotient and submodule in $\mathcal{O}^{\mathfrak{p}}$, respectively. Denote by $W_{\mathfrak{p}}$ the parabolic subgroup of W corresponding to the Levi factor of $\mathfrak{p}$.

Proposition 4.11. *For* $\mu \in \mathfrak{h}^*$ *we have* $L(\mu) \in \mathcal{O}^{\mathfrak{p}}$ *if and only if* $w \cdot \mu < \mu$ *for all* $w \in W_{\mathfrak{p}}$, $w \neq e$.

In other words, $L(\mu) \in \mathcal{O}^{\mathfrak{p}}$ if and only if μ is $W_{\mathfrak{p}}$-dominant. In particular, if λ is dominant, integral and regular, then simple modules in $\mathcal{O}^{\mathfrak{p}}_\lambda$ have the form $L(w \cdot \lambda)$, where w is the shortest coset representative for some coset from $W_{\mathfrak{p}} \backslash W$. For $\mu \in \mathfrak{h}^*$ we set

$$L^{\mathfrak{p}}(\mu) = \mathrm{Z}_{\mathfrak{p}} L(\mu), \quad P^{\mathfrak{p}}(\mu) = \mathrm{Z}_{\mathfrak{p}} P(\mu), \quad \Delta^{\mathfrak{p}}(\mu) = \mathrm{Z}_{\mathfrak{p}} \Delta(\mu).$$

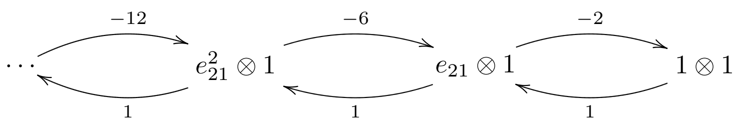
Modules $\Delta^{\mathfrak{p}}(\mu)$ are called *parabolic Verma modules* and can be alternatively described using parabolic induction from a simple finite-dimensional $\mathfrak{p}$-module.

For $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ the category $\mathcal{O}^{\mathfrak{p}}_\lambda$ is equivalent to the category of modules over the quotient algebra $B^{\mathfrak{p}}_\lambda = B_\lambda / B_\lambda e B_\lambda$, where e is a maximal idempotent annihilating $\mathcal{O}^{\mathfrak{p}}_\lambda$. The duality $\star$ preserves $\mathcal{O}^{\mathfrak{p}}$ and hence $B^{\mathfrak{p}}_\lambda = (B^{\mathfrak{p}}_\lambda)^{\mathrm{op}}$. We set

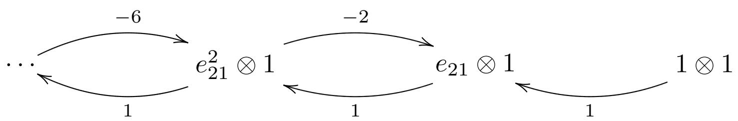
$$\nabla^{\mathfrak{p}}(\mu) = \Delta^{\mathfrak{p}}(\mu)^\star, \quad I^{\mathfrak{p}}(\mu) = P^{\mathfrak{p}}(\mu)^\star.$$

Finally, the algebra $B^{\mathfrak{p}}_\lambda$ is quasi-hereditary with parabolic Verma modules being standard modules with respect to this structure, see [RC]. We denote by $T^{\mathfrak{p}}(\mu)$ the corresponding indecomposable tilting modules and by $T^{\mathfrak{p}}_\lambda$ the corresponding characteristic tilting module. Note that these cannot be obtained from the corresponding tilting modules in $\mathcal{O}$ using Zuckerman functors naïvely. Similarly to B_λ, the algebra $B^{\mathfrak{p}}_\lambda$ is Ringel self-dual. One important difference with the usual category $\mathcal{O}$ is that parabolic Verma modules might have non-simple socle and that the dimension of the homomorphism space between different parabolic Verma modules might be bigger than two.

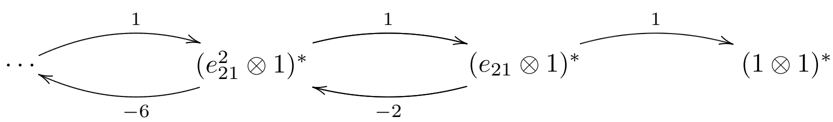
4.7. **$\mathfrak{gl}_2$-example.** For $n = 2$ every singular block is semi-simple (as W contains only two elements). Note that we have $\rho = \frac{1}{2}(1, -1)$. The regular block $\mathcal{O}_0$ contains two simple modules, $L(0,0)$ and $L(-1,1) = \Delta(-1,1) = \Delta(-1,1)^* = T(-1,1)$. The module $L(0,0)$ is the trivial module. The action of $\mathfrak{g}$ on the module $L(-1,1)$, considered as a Verma module, is given by the following picture (here left arrows represent the action of e_{21}, right arrows represent the action of e_{12}, and numbers are coefficients):



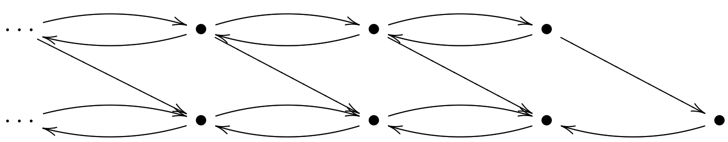
The Verma module $\Delta(0,0) = P(0,0)$ is given by the following picture (no right arrow means that the corresponding coefficient is zero):



The dual Verma module $\Delta(0,0)^\star$ is given by the following picture:



In this small example there is only one indecomposable module left, namely the projective module $P(-1,1)$. One can choose a basis in $P(-1,1)$ (given by bullets below) such that the action of e_{12} and e_{21} in this basis can be given by the following picture (see [Ma7, Chapter 3] for explicit coefficients):



In particular, we see that $\Delta(0) \subset P(-1,1)$ and the cokernel of this map is isomorphic to $\Delta(-1,1)$. This means that $P(-1,1) = T(0) = P(-1,1)^*$. The algebra B_0

is given by the following quiver and relations:

$$s \underset{b}{\overset{a}{\rightleftarrows}} e\,, \qquad ab = 0. \tag{4.1}$$

Here the vertex e corresponds to the simple $L(0,0)$ and the vertex s corresponds to the simple $L(-1,1)$. Note that all indecomposable B_0-modules are uniserial. Abbreviating B_0-simples by the notation for the corresponding vertexes, we obtain the following unique Loewy filtrations of indecomposable B_0-modules:

$L(0,0)$	$L(-1,1)$	$\Delta(0,0)$	$\nabla(0,0)$	$P(-1,1)$
e	s	e s	s e	s e s

The only non-Borel parabolic subalgebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself. Hence $\mathcal{O}_0^{\mathfrak{g}}$ is the semi-simple category with simple module $L(0,0)$.

5. Category $\mathcal{O}$: projective and shuffling functors

5.1. **Projective functors.** Recall that for any two $\mathfrak{g}$-modules X and Y the vector space $X \otimes Y$ has the structure of a $\mathfrak{g}$-module given by $g(x \otimes y) = g(x) \otimes y + x \otimes g(y)$ for all $g \in \mathfrak{g}$, $x \in X$ and $y \in Y$. Hence for every $\mathfrak{g}$-module V we have the endofunctor $V \otimes_{\mathbb{C}} {}_-$ on the category of $\mathfrak{g}$-modules. If V is finite dimensional, the functor $V \otimes_{\mathbb{C}} {}_-$ preserves $\mathcal{O}$.

Definition 5.1. A functor $\theta : \mathcal{O} \to \mathcal{O}$ (or $\theta : \mathcal{O}_\lambda \to \mathcal{O}_{\lambda'}$) is called *projective* if it is isomorphic to a direct summand of some $V \otimes_{\mathbb{C}} {}_-$, where V is finite dimensional.

For example, the identity functor is a projective functor (it is isomorphic to the tensoring with the one-dimensional $\mathfrak{g}$-module). Projective functors appeared already in [BGG1, Ja1]. Indecomposable projective functors are classified by:

Theorem 5.2 ([BG])**.** *Let λ and λ' be dominant and integral.*

(*a*) *For every W_λ-antidominant $\mu \in W \cdot \lambda'$ there is a unique indecomposable projective functor $\theta_{\lambda,\mu}$ such that $\theta_{\lambda,\mu}\,\Delta(\lambda) = P(\mu)$.*
(*b*) *Every indecomposable projective functor from $\mathcal{O}_\lambda$ to $\mathcal{O}_{\lambda'}$ is isomorphic to $\theta_{\lambda,\mu}$ for some W_λ-antidominant $\mu \in W \cdot \lambda'$.*

Theorem 5.2 implies that an indecomposable projective functor θ is completely determined by its value $\theta\,\Delta(\lambda)$ on the corresponding dominant Verma module $\Delta(\lambda)$. Moreover, as $\theta\,\Delta(\lambda)$ is projective and projective modules form a basis of $[\mathcal{O}_{\lambda'}]$ (as $\mathcal{O}_{\lambda'}$, being quasi-hereditary, has finite global dimension), the functor θ is already uniquely determined by $[\theta\,\Delta(\lambda)]$.

Another consequence is that in case λ is regular, indecomposable projective endofunctors of $\mathcal{O}_\lambda$ are in a natural bijection with W. For $w \in W$ we will denote by θ_w the indecomposable projective functor such that $\theta_w\,\Delta(\lambda) = P(w \cdot \lambda)$. Here are some basic general properties of projective functors (see e.g. [BG]):

Proposition 5.3. (*a*) *Any direct summand of a projective functor is a projective functor.*
(*b*) *Any direct sum of projective functors is a projective functor.*
(*c*) *Any composition of projective functors is a projective functor.*
(*d*) *Every projective functor is both left and right adjoint to some other projective functor.*
(*e*) *Projective functors are exact.*
(*f*) *Projective functors preserve the additive category of projective modules and the additive category of injective modules.*

(*g*) *Projective functors preserve both* $\mathcal{F}(\Delta)$ *and* $\mathcal{F}(\nabla)$ *and hence also the additive category of tilting modules.*
(*h*) *Projective functors commute with* $\star$.

For projective endofunctors of $\mathcal{O}_\lambda$ the claim of Proposition 5.3(d) can be made more explicit:

Proposition 5.4. *Let* λ *be regular, integral and dominant. Then for* $w \in W$ *the functor* θ_w *is both left and right adjoint to* $\theta_{w^{-1}}$.

5.2. **Translations through walls.** Let λ be regular, integral and dominant. For a simple reflection s the projective endofunctor θ_s of $\mathcal{O}_\lambda$ is called *translation through the* s*-wall.* Let μ be integral and dominant such that $W_\mu = \{e, s\}$. Then

$$\theta_s \cong \theta_{\mu, s\cdot\lambda} \circ \theta_{\lambda,\mu}. \tag{5.1}$$

The functor $\theta_{\mu,s\cdot\lambda}$ is called *translation out of the* s*-wall* and is both left and right adjoint to the functor $\theta_{\lambda,\mu}$, the latter being called *translation to the* s*-wall.* These "smallest" translations θ_s have the following properties:

Proposition 5.5. *Let* λ *be regular, integral and dominant.*
(*a*) *Let* $w \in W$ *and* $w = s_1 s_2 \cdots s_k$ *be a reduced decomposition. Then there is a unique direct summand of* $\theta_{s_k}\theta_{s_{k-1}}\cdots\theta_{s_1}$ *isomorphic to* θ_w. *All other direct summands of* $\theta_{s_k}\theta_{s_{k-1}}\cdots\theta_{s_1}$ *are isomorphic to* θ_x *for* $x < w$.
(*b*) *For any simple reflection* $s \in S$ *and any* $w \in W$ *there are exact sequences*

$$\begin{array}{ll} 0 \to \Delta(w\cdot\lambda) \to \theta_s\Delta(w\cdot\lambda) \to \Delta(ws\cdot\lambda) \to 0, & ws > w; \\ 0 \to \Delta(ws\cdot\lambda) \to \theta_s\Delta(w\cdot\lambda) \to \Delta(w\cdot\lambda) \to 0, & ws < w. \end{array}$$

(*c*) $\theta_s \circ \theta_s \cong \theta_s \oplus \theta_s$.
(*d*) *If* s *and* t *are commuting simple reflections, then* $\theta_s \circ \theta_t \cong \theta_t \circ \theta_s$.
(*e*) *If* s *and* t *are simple reflections such that* $sts = tst$, *then*

$$(\theta_s \circ \theta_t \circ \theta_s) \oplus \theta_t \cong (\theta_t \circ \theta_s \circ \theta_t) \oplus \theta_s.$$

Idea of the proof. Apply functors to the dominant Verma and compute the images of the results in the Grothendieck group. □

For a regular, integral and dominant λ denote by $\mathscr{S}_\lambda$ the category of projective endofunctors of $\mathcal{O}_\lambda$. The category $\mathscr{S}_\lambda$ is additive and monoidal (the monoidal structure is given by composition of functors). Hence $\mathscr{S}_\lambda$ can be viewed as the endomorphism category of the object in a 2-category with one object. Recall that $W \cong \mathbb{S}_n$. The following results provides a categorification of the integral group ring $\mathbb{Z}[\mathbb{S}_n]$:

Corollary 5.6 (Categorification of the integral group ring of $\mathbb{S}_n$)**.** *The map*

$$\begin{array}{ccc} \mathbb{Z}[W] & \to & [\mathscr{S}_\lambda] \\ e+s & \mapsto & [\theta_s] \end{array}$$

induces an anti-isomorphism of unital rings.

Idea of the proof. The ring $\mathbb{Z}[W]$ is generated, as a unital ring, by elements $e+s$, where s is a simple reflection. The defining relations of $\mathbb{Z}[W]$ in this basis are:

$$(e+s)^2 = e+s; \quad (e+s)(e+t) = (e+t)(e+s) \text{ if } st = ts;$$
$$(e+s)(e+t)(e+s) + (e+t) = (e+t)(e+s)(e+t) + (e+s) \text{ if } sts = tst.$$

By Proposition 5.5(c)-(e), the elements $[\theta_s]$ satisfy these relations and hence the map $e+s \mapsto [\theta_s]$ induces a ring homomorphism. It is surjective as translations through walls generate $\mathscr{S}_\lambda$ by Proposition 5.5(a). Bijectivity now follows by comparing ranks. □

From Corollary 5.6 it follows that the set $\{[\theta_w] : w \in W\}$ is a new basis of $\mathbb{Z}[W]$. Later we will identify this basis as the *Kazhdan-Lusztig* basis. Note that existence of this basis is a bonus of our categorification result. From Proposition 5.5(b) it follows that the action of projective functors on $\mathcal{O}_\lambda$ categorifies the right regular representation of $\mathbb{S}_n$:

Proposition 5.7 (Categorification of the right regular $\mathbb{S}_n$-module)**.** (a) *There is a unique isomorphism of abelian groups $\varphi : \mathbb{Z}[\mathbb{S}_n] \to [\mathcal{O}_\lambda]$ such that $\varphi(w) = [\Delta(w \cdot \lambda)]$.*

(b) *For any $s \in S$ the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{S}_n] & \xrightarrow{\cdot(e+s)} & \mathbb{Z}[\mathbb{S}_n] \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ [\mathcal{O}_\lambda] & \xrightarrow{[\theta_s]\cdot} & [\mathcal{O}_\lambda] \end{array}$$

The decategorification of the action of projective functors on $\mathcal{O}_\lambda$ can thus be depicted as follows:

$$\mathcal{O}_\lambda \circlearrowleft_{\mathscr{S}_\lambda} \qquad \mapsto \qquad [\mathcal{O}_\lambda] \circlearrowleft_{[\mathscr{S}_\lambda]} \qquad \cong \qquad \mathbb{Z}[\mathbb{S}_n] \circlearrowright_{\mathbb{Z}[\mathbb{S}_n]}$$

5.3. **Description via Harish-Chandra bimodules.** Projective functors, being right exact, can be described, by the general nonsense, as tensoring with certain bimodules. The identity functor is of course isomorphic to the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} {}_-$. Hence, for a finite dimensional module V, the functor $V \otimes_{\mathbb{C}} {}_-$ is isomorphic to the functor $V \otimes_{\mathbb{C}} U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} {}_-$. However, the bimodule $V \otimes_{\mathbb{C}} U(\mathfrak{g})$ is "too big". On the other hand, if we fix dominant λ, then any module in $\mathcal{O}_\lambda$ is annihilated by $(\mathrm{Ker}\chi_\lambda)^k$ for some fixed big enough k (since every module is a quotient of a projective module, we have only finitely many indecomposable projectives in $\mathcal{O}_\lambda$, every projective has finite length, and any simple is annihilated by $\mathrm{Ker}\chi_\lambda$). Therefore, restricted to $\mathcal{O}_\lambda$, the functors $U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} {}_-$ and $U(\mathfrak{g})/(\mathrm{Ker}\chi_\lambda)^k \otimes_{U(\mathfrak{g})} {}_-$ are isomorphic. The bimodule $M = U(\mathfrak{g})/(\mathrm{Ker}\chi_\lambda)^k$ is now reasonably "small" in the following sense: Under the adjoint action of $\mathfrak{g}$ the bimodule M decomposes into a direct sum of simple finite-dimensional $\mathfrak{g}$-modules, each occurring with finite multiplicity. A finitely generated $\mathfrak{g}$-bimodule satisfying this condition is called a *Harish-Chandra* bimodule. The category of Harish-Chandra bimodules is denoted by $\mathcal{H}$ (see [Ja2, Kapitel 6] for details).

We have the following decomposition of $\mathcal{H}$ with respect to the action of $Z(\mathfrak{g})$:

$$\mathcal{H} = \bigoplus_{\lambda,\mu \in \mathfrak{h}^*_{\mathrm{dom}}} {}_\lambda\mathcal{H}_\mu,$$

where ${}_\lambda\mathcal{H}_\mu$ denotes the full subcategory of $\mathcal{H}$, which consists of all bimodules M such that for any $v \in M$ we have $(\mathrm{Ker}\chi_\lambda)^k v = 0$ and $v(\mathrm{Ker}\chi_\mu)^k = 0$ for $k \gg 0$. For $k, l \in \{\infty, 1, 2, \dots\}$ denote by ${}^k_\lambda\mathcal{H}^l_\mu$ the full subcategory of ${}_\lambda\mathcal{H}_\mu$ consisting of all M satisfying $(\mathrm{Ker}\chi_\lambda)^k M = 0$ and $M(\mathrm{Ker}\chi_\mu)^l = 0$. Then we have:

Theorem 5.8 ([BG])**.** *Let λ' be integral and dominant and λ be integral, dominant and regular. Then tensoring with $\Delta(\lambda)$ induces an equivalence ${}^\infty_{\lambda'}\mathcal{H}^1_\lambda \cong \mathcal{O}_{\lambda'}$.*

Under the equivalence from Theorem 5.8, indecomposable projective functors from $\mathcal{O}_\lambda$ to $\mathcal{O}_{\lambda'}$ correspond to indecomposable Harish-Chandra bimodules. It follows that every $\theta_{\lambda,\mu} : \mathcal{O}_\lambda \to \mathcal{O}_{\lambda'}$ is isomorphic to tensoring with the corresponding indecomposable projective object from ${}^{\infty}_{\lambda'}\mathcal{H}^{l}_{\lambda}$ for appropriate l. The same description is true in the case of singular λ, though in this case ${}^{\infty}_{\lambda'}\mathcal{H}^{1}_{\lambda}$ is isomorphic only to a certain quotient of $\mathcal{O}_{\lambda'}$.

Let λ be integral, dominant and regular. Then the category ${}^{\infty}_{\lambda}\mathcal{H}^{\infty}_{\lambda}$ has the natural monoidal structure given by tensor product of bimodules. The disadvantage of ${}^{\infty}_{\lambda}\mathcal{H}^{\infty}_{\lambda}$ is that this category does not have projective modules. The category ${}^{\infty}_{\lambda}\mathcal{H}^{\infty}_{\lambda}$ can be described as a *deformation* of the category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\lambda}$ and can be realized via the so-called *thick category* $\tilde{\mathcal{O}}$, defined by weakening the condition of $\mathfrak{h}$-diagonalizability to the condition of local finiteness of the $U(\mathfrak{h})$-action.

5.4. **Shuffling functors.** As we have seen in Corollary 5.6, the projective functors θ_s are categorical analogues of the elements $e+s\in\mathbb{Z}[\mathbb{S}_n]$. A natural question is: what are analogues of simple reflections? As $s=(e+s)-e$, and the categorical analogue of e is the identity functor θ_e, to answer this question we have to "subtract" θ_e from θ_s in some way. Naïvely this is not possible as θ_s is an indecomposable functor. But, recall from Subsection 2.3 that we have another natural way to interpret negative coefficients: as shifts in homological position. This suggests to try to realize s as the cone of some morphism between θ_e and θ_s. And it turns out that there is a very natural candidate for this morphism.

By (5.1), the functor θ_s is the composition of two functors, both left and right adjoint to each other. Let $\mathrm{adj}_s:\theta_e\to\theta_s$ and $\overline{\mathrm{adj}}_s:\theta_s\to\theta_e$ be the corresponding adjunction morphisms. Denote by C_s the cokernel of adj_s and by K_s the kernel of $\overline{\mathrm{adj}}_s$. The functor C_s is called the *shuffling functor* and the functor K_s is called the *coshuffling functor*. These functors appeared first in [Ca] and then were studied in more details in [Ir2] and further in [MS1]. Here are some basic properties of (co)shuffling functors (see e.g. [MS1]):

Proposition 5.9. *Let $s\in W$ be a simple reflection and λ be dominant integral and regular.*

(*a*) *The pair $(\mathrm{C}_s,\mathrm{K}_s)$ is adjoint.*
(*b*) *There is an isomorphism $\mathrm{K}_s\cong\star\circ\mathrm{C}_s\circ\star$.*
(*c*) *For $w\in W$ such that $ws>w$ we have $\mathrm{C}_s\Delta(w\cdot\lambda)\cong\Delta(ws\cdot\lambda)$.*
(*d*) *For $w\in W$ such that $ws<w$ there is an exact sequence*

$$0\to\mathrm{Coker}_w\to\mathrm{C}_s\Delta(w\cdot\lambda)\to\Delta(w\cdot\lambda)\to 0,$$

where Coker_w denotes the cokernel of the natural inclusion $\Delta(w\cdot\lambda)\hookrightarrow\Delta(ws\cdot\lambda)$.
(*e*) *We have $\mathcal{L}_i\mathrm{C}_s\Delta(w\cdot\lambda)=0$ for all $w\in W$ and all $i>0$, in particular, the restriction of C_s to $\mathcal{F}(\Delta)$ is exact.*
(*f*) $\mathcal{L}_i\mathrm{C}_s=\begin{cases}\mathrm{Ker}(\mathrm{adj}_s), & i=1;\\ 0, & i>1.\end{cases}$
(*g*) *$\mathcal{L}\mathrm{C}_s$ is an autoequivalence of $\mathcal{D}^b(\mathcal{O}_\lambda)$ with inverse $\mathcal{R}\mathrm{K}_s$.*

Combining Proposition 5.9(c)-(d) we have the following corollary, which says that C_s is a "good" candidate for the role of categorification of s.

Corollary 5.10. *For any $w\in W$ and any simple reflection $s\in W$ we have the following equality: $[\mathrm{C}_s\Delta(w\cdot\lambda)]=[\Delta(ws\cdot\lambda)]$.*

Since C_s is only right exact, to be able to categorify anything using it we have to consider the endofunctor $\mathcal{L}\mathrm{C}_s$ of $\mathcal{D}^b(\mathcal{O}_\lambda)$. The latter is invertible by Proposition 5.9(g). Unfortunately, from Proposition 5.9(f) it follows that $\mathcal{L}\mathrm{C}_s\circ\mathcal{L}\mathrm{C}_s\not\cong\mathrm{Id}$. Therefore the action of $\mathcal{L}\mathrm{C}_s$ of $\mathcal{D}^b(\mathcal{O}_\lambda)$ cannot be used to categorify $\mathbb{S}_n$. However, we have the following:

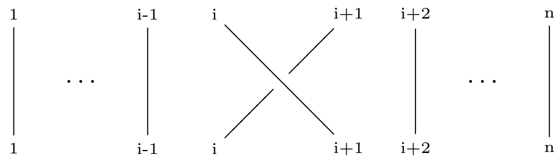
Proposition 5.11. *Let $s,t\in W$ be simple reflections.*

(*a*) *If* $st = ts$*, then we have both* $\mathrm{C}_s \circ \mathrm{C}_t \cong \mathrm{C}_t \circ \mathrm{C}_s$ *and* $\mathcal{L}\mathrm{C}_s \circ \mathcal{L}\mathrm{C}_t \cong \mathcal{L}\mathrm{C}_t \circ \mathcal{L}\mathrm{C}_s$*.*
(*b*) *If* $sts = sts$*, then we have both* $\mathrm{C}_s \circ \mathrm{C}_t \circ \mathrm{C}_s \cong \mathrm{C}_t \circ \mathrm{C}_s \circ \mathrm{C}_t$ *and* $\mathcal{L}\mathrm{C}_s \circ \mathcal{L}\mathrm{C}_t \circ \mathcal{L}\mathrm{C}_s \cong \mathcal{L}\mathrm{C}_t \circ \mathcal{L}\mathrm{C}_s \circ \mathcal{L}\mathrm{C}_t$*.*

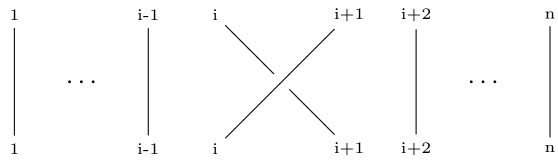
Proposition 5.11 says that shuffling functors satisfy braid relations and hence define a weak action of the braid group $\mathbb{B}_n$ on $\mathcal{D}^b(\mathcal{O}_\lambda)$. From Corollary 5.10 it follows that the decategorification of this action gives a representation of $\mathbb{S}_n$ (which is isomorphic to the right regular representation of $\mathbb{S}_n$ by Proposition 5.7). This action can be understood in terms of 2-categories. Rouquier defines in [Ro1] a 2-category $\mathfrak{B}_n$ whose decategorification is isomorphic to $\mathbb{B}_n$ and shows that the above action of shuffling functors on $\mathcal{D}^b(\mathcal{O}_\lambda)$ can be considered as a 2-representation of $\mathfrak{B}_n$.

5.5. **Singular braid monoid.** To this end we have both projective and (co)shuffling functors acting on $\mathcal{D}^b(\mathcal{O}_\lambda)$. The former categorify an action of $\mathbb{Z}[\mathbb{S}_n]$ and the latter categorify an action of $\mathbb{B}_n$. A natural question is: What kind of object is categorified if we consider all these functors together? It turns out that the answer is: the singular braid monoid $\mathcal{S}\mathbb{B}_n$. This monoid is defined in [Ba, Bi] as the monoid generated by the following elements:

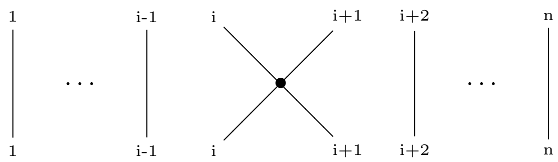
$\sigma_{\mathrm{i}} =$



$\sigma_{\mathrm{i}}^{-1} =$



$\tau_{\mathrm{i}} =$



These generators satisfy the following relations:

$$
\begin{array}{rl}
\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, & \text{for all } i; \\
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \text{for all } i; \\
\sigma_i\sigma_j = \sigma_j\sigma_i, & \text{if } |i-j| > 1; \\
\tau_i\sigma_j\sigma_i = \sigma_j\sigma_i\tau_j, & \text{if } |i-j| = 1; \\
\sigma_i\tau_j = \tau_j\sigma_i, & \text{if } |i-j| \neq 1; \\
\tau_i\tau_j = \tau_j\tau_i, & \text{if } |i-j| > 1.
\end{array}
$$

The following theorem is proved in [MS2]:

Theorem 5.12 (Weak categorification of $\mathcal{S}\mathbb{B}_n$)**.** *Let* λ *be regular, integral and dominant. Then, mapping* $\sigma_i \mapsto \mathcal{L}\mathrm{C}_{s_i}$*,* $\sigma_i^{-1} \mapsto \mathcal{R}\mathrm{K}_{s_i}$ *and* $\tau_i \mapsto \theta_{s_i}$*, for* $i = 1, 2, \dots, n-1$*, defines a weak action of* $\mathcal{S}\mathbb{B}_n$ *on* $\mathcal{D}^b(\mathcal{O}_\lambda)$*.*

5.6. **$\mathfrak{gl}_2$-example.** The action of projective, (co)shuffling and the corresponding homology functors on the principal block of $\mathfrak{gl}_2$ is given by the following table (notation as in Subsection 4.7):

M	e	s	e s	s e	s e s
$\theta_s\, M$	0	s e s	s e s	s e s	s e s $\oplus$ s e s
$\mathrm{C}_s\, M$	0	s e	s	s e	s e s
$\mathrm{K}_s\, M$	0	e s	e s	s	s e s
$\mathcal{L}_1\mathrm{C}_s\, M$	e	0	0	e	0
$\mathcal{R}_1\mathrm{K}_s\, M$	e	0	e	0	0

6. Category $\mathcal{O}$: twisting and completion

6.1. **Zuckerman functors.** As mentioned in Subsection 4.6, for every parabolic $\mathfrak{p} \subset \mathfrak{g}$ we have the corresponding Zuckerman functor $\mathrm{Z}_\mathfrak{p} : \mathcal{O} \to \mathcal{O}^\mathfrak{p}$ and dual Zuckerman functor $\hat{\mathrm{Z}}_\mathfrak{p} : \mathcal{O} \to \mathcal{O}^\mathfrak{p}$, which are the left and the right adjoints of the natural inclusion $\mathcal{O}^\mathfrak{p} \hookrightarrow \mathcal{O}$. It is easy to see that $\hat{\mathrm{Z}}_\mathfrak{p} \cong \star \circ \mathrm{Z}_\mathfrak{p} \circ \star$. As every $\mathcal{O}^\mathfrak{p}$ is stable under the action of projective functors, we have the following:

Proposition 6.1. *Both $\mathrm{Z}_\mathfrak{p}$, $\hat{\mathrm{Z}}_\mathfrak{p}$ and the corresponding derived functors commute with all projective functors.*

If the semi-simple part of the Levi factor of $\mathfrak{p}$ is isomorphic to $\mathfrak{sl}_2$ and s is the corresponding simple reflection, we will denote the corresponding Zuckerman functors simply by Z_s and $\hat{\mathrm{Z}}_s$, respectively. For the derived versions of Zuckerman functors we have:

Proposition 6.2 ([EW]). *Let s be a simple reflection.*

(a) *We have $\mathcal{L}_i\mathrm{Z}_s = 0$, $i > 2$.*
(b) *We have $\mathcal{L}_2\mathrm{Z}_s = \hat{\mathrm{Z}}_s$.*
(c) *For the functor $\mathrm{Q}_s := \mathcal{L}_1\mathrm{Z}_s$ we have $\mathrm{Q}_s \cong \star \circ \mathrm{Q}_s \circ \star$.*
(d) *We have $\mathcal{L}\mathrm{Z}_s[-1] \cong \mathcal{R}\hat{\mathrm{Z}}_s[1]$.*

Example 6.3. The action of Z_s, $\hat{\mathrm{Z}}_s$ and Q_s on the principal block of the category $\mathcal{O}$ for the algebra $\mathfrak{gl}_2$ is given in the following table:

M	e	s	e s	s e	s e s
$\mathrm{Z}_s\, M$	e	0	e	0	0
$\hat{\mathrm{Z}}_s\, M$	e	0	0	e	0
$\mathrm{Q}_s\, M$	0	e	0	0	0

6.2. **Twisting functors.** For $i = 1, 2, \dots, n-1$ denote by U_i the localization of the algebra $U(\mathfrak{g})$ with respect to the multiplicative set $\{e_{i+1i}^k : k \in \mathbb{N}\}$. If $\{x_1 = e_{i+1i}, x_2, \dots, x_k\}$ is a basis of $\mathfrak{g}$, then U_i has a *PBW-type* basis consisting of all monomials of the form $x_1^{m_1}x_2^{m_2}\cdots x_k^{m_k}$, where $m_1 \in \mathbb{Z}$ and $m_2, \dots, m_k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, $U(\mathfrak{g})$ is a subalgebra of U_i is the natural way. Denote by $\mathtt{B}_i$

the $U(\mathfrak{g})$-$U(\mathfrak{g})$ bimodule $U_i/U(\mathfrak{g})$. Let α_i be an automorphism of $\mathfrak{g}$ corresponding to the simple reflection s_i. Consider the bimodule ${}^{\alpha_i}\mathtt{B}_i$ in which the left action is twisted by α_i and define the endofunctor T_i of $\mathfrak{g}$-Mod as tensoring with ${}^{\alpha_i}\mathtt{B}_i$. The functor T_i is called *twisting functor*. Twisting functors appeared first in [Ar1], were used in [So3, Ar2], in particular, to prove Ringel self-duality of $\mathcal{O}$, and were studied in [AS] and [KhMa2] in detail.

The action of T_i on a module M can be understood as the composition of the following three operations:

- invert the action of the element e_{i+1i}, obtaining the localized module M_1;
- factor out the canonical image of M in M_1, obtaining the quotient M_2;
- twist the action of $U(\mathfrak{g})$ on M_2 by α_i, obtaining $\mathrm{T}_i\, M$.

Twisting functors have the following basic properties (see [AS]):

Proposition 6.4. *Let $i \in \{1, 2, \dots, n-1\}$.*

(*a*) *The functor T_i preserves both $\mathcal{O}$ and $\mathcal{O}_\lambda$ for any $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$.*
(*b*) *The functor T_i is right exact.*
(*c*) *The functor T_i is left adjoint to the functor $\mathrm{G}_i := \star \circ \mathrm{T}_i \circ \star$.*
(*d*) *For any finite-dimensional $\mathfrak{g}$-module V the functors $V \otimes_{\mathbb{C}} {}_-$ and T_i commute. In particular, T_i commutes will all projective functors.*

It turns out that twisting functors satisfy braid relations (see e.g. [KhMa2]):

Proposition 6.5. (*a*) *For $i, j \in \{1, 2, \dots, n-1\}$, $i \neq j \pm 1$, we have $\mathrm{T}_i \circ \mathrm{T}_j \cong \mathrm{T}_j \circ \mathrm{T}_i$.*
(*b*) *For $i \in \{1, 2, \dots, n-2\}$ we have $\mathrm{T}_i \circ \mathrm{T}_{i+1} \circ \mathrm{T}_i \cong \mathrm{T}_{i+1} \circ \mathrm{T}_i \circ \mathrm{T}_{i+1}$.*

For $w \in W$ choose a reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ of w and define the *twisting functor* T_w as follows: $\mathrm{T}_w := \mathrm{T}_{i_1} \circ \mathrm{T}_{i_2} \circ \dots \circ \mathrm{T}_{i_k}$. From Proposition 6.5 it follows that T_w does not depend (up to isomorphism) on the choice of the reduced decomposition. In the following claim we collected some combinatorial properties of twisting functors (see [AS]):

Proposition 6.6. (*a*) *For any $\mu \in \mathfrak{h}^*$ and any $s \in S$ we have*

$$\mathrm{T}_s\, \nabla(\mu) = \begin{cases} \nabla(s \cdot \mu), & \mu < s \cdot \mu; \\ \nabla(\mu), & \textit{otherwise}. \end{cases}$$

(*b*) *For all $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ and $w \in W$ we have $[\mathrm{T}_w\, \Delta(\lambda)] = [\Delta(w \cdot \lambda)]$.*

The most important properties of twisting functors are the ones connected to their extensions to the derived category (see [AS, MS2]):

Proposition 6.7. *Let $s \in S$ and $w \in W$.*

(*a*) *We have $\mathcal{L}_i \mathrm{T}_w \Delta(\mu) = 0$ for all $\mu \in \mathfrak{h}^*$ and $i > 0$.*
(*b*) *We have $\mathcal{L}_i \mathrm{T}_s = 0$ for all $i > 1$.*
(*c*) *We have $\mathcal{L}_1 \mathrm{T}_s = \hat{\mathrm{Z}}_s$.*
(*d*) *If $x, y \in W$ are such that $\mathfrak{l}(xy) = \mathfrak{l}(x) + \mathfrak{l}(y)$, then $\mathcal{L}\mathrm{T}_{xy} \cong \mathcal{L}\mathrm{T}_x \circ \mathcal{L}\mathrm{T}_y$.*
(*e*) *The functor $\mathcal{L}\mathrm{T}_s$ is a self-equivalence of $\mathcal{D}^b(\mathcal{O})$ with inverse $\mathcal{R}\mathrm{G}_s$.*

Another important property of twisting functors is the following (see [AS, KhMa2, MS2]):

Proposition 6.8. *For any simple reflection s there is an exact sequence of functors as follows: $\mathrm{Q}_s \hookrightarrow \mathrm{T}_s \to \mathrm{Id}$.*

Combining Proposition 6.5 and Proposition 6.7(d), we obtain that the functors $\mathcal{L}\mathrm{T}_i$, $i = 1, 2, \dots, n-1$, satisfy braid relations and hence define a weak action of the braid group on $\mathcal{D}^b(\mathcal{O})$. Decategorifying this action we obtain:

Proposition 6.9 (Naïve categorification of the left regular $\mathbb{S}_n$-module)**.** *Let $\lambda\in\mathfrak{h}^*$ be dominant, integral and regular. Then there is a unique isomorphism $\overline{\varphi}:\mathbb{Z}[\mathbb{S}_n]\to[\mathcal{D}^b(\mathcal{O}_\lambda)]$ such that $\overline{\varphi}(w)=[\Delta(w\cdot\lambda)]$ for any $w\in\mathbb{S}_n$. Furthermore, for any $w\in\mathbb{S}_n$ the following diagram commutes:*

$$\begin{array}{ccc}\mathbb{Z}[\mathbb{S}_n] & \xrightarrow{w\cdot} & \mathbb{Z}[\mathbb{S}_n]\\ \downarrow{\scriptstyle\overline{\varphi}} & & \downarrow{\scriptstyle\overline{\varphi}}\\ [\mathcal{D}^b(\mathcal{O}_\lambda)] & \xrightarrow{[\mathrm{T}_w]\cdot} & [\mathcal{D}^b(\mathcal{O}_\lambda)]\end{array},$$

where φ is as in Proposition 5.7.

The above categorification is only a naïve one since $\mathcal{L}\mathrm{T}_i\circ\mathcal{L}\mathrm{T}_i\neq\mathrm{Id}$ (for example because of Propositions 6.7(c)). Combining Propositions 5.7 and 6.9 with Proposition 6.4(d) we obtain a naïve categorification of the regular $\mathbb{Z}[\mathbb{S}_n]$-bimodule.

Example 6.10. The action of T_s on the principal block of the category $\mathcal{O}$ for the algebra $\mathfrak{gl}_2$ is given by the following table:

M	e	s	e s	s e	s e s
$\mathrm{T}_s\,M$	0	s e	s	s e	s e s

6.3. **Completion functors.** Let λ' be integral and dominant, and λ be integral, dominant and regular. According to Theorem 5.8, the category $\mathcal{O}_{\lambda'}$ is equivalent to the category ${}^{\infty}_{\lambda'}\mathcal{H}^1_\lambda$ of Harish-Chandra bimodules. The latter is a subcategory of the category ${}^{\infty}_{\lambda'}\mathcal{H}^\infty_\lambda$. The category ${}^{\infty}_{\lambda'}\mathcal{H}^\infty_\lambda$ is very symmetric. In particular, it has both the usual (left) projective functors $\theta^l_{\lambda',\mu}$, as defined in Subsection 5.1, and the right projective functors $\theta^r_{\lambda,\mu}$, defined similarly using tensoring of Harish-Chandra bimodules with finite dimensional $\mathfrak{g}$-modules on the right. Since λ is assumed to be regular, we can index indecomposable right projective functors on ${}^{\infty}_{\lambda'}\mathcal{H}^\infty_\lambda$ by elements from W and denote them by θ^r_w, $w\in W$. Unfortunately, the functors θ^r_w do not preserve the subcategory ${}^{\infty}_{\lambda'}\mathcal{H}^1_\lambda$ in general. However, similarly to the left projective functors, for every simple reflection s there are adjunction morphisms

$$\mathrm{adj}^r_s:\theta^r_e\to\theta^r_s;\qquad \overline{\mathrm{adj}}^r_s:\theta^r_s\to\theta^r_e.$$

Let C^r_s and K^r_s denote the cokernel of the first morphism and the kernel of the second morphism, respectively. Similarly to Proposition 5.9(a), we have the pair $(\mathrm{C}^r_s,\mathrm{K}^r_s)$ of adjoint functors on ${}^{\infty}_{\lambda'}\mathcal{H}^\infty_\lambda$.

It is easy to show that both C^r_s and K^r_s preserve ${}^{\infty}_{\lambda'}\mathcal{H}^1_\lambda$ and hence we can restrict the adjoint pair $(\mathrm{C}^r_s,\mathrm{K}^r_s)$ to ${}^{\infty}_{\lambda'}\mathcal{H}^1_\lambda$. Via the equivalence from Theorem 5.8 this defines a pair of adjoint functors on $\mathcal{O}_{\lambda'}$. Let us denote by G_s the endofunctor of $\mathcal{O}_{\lambda'}$, which corresponds to K^r_s under the equivalence of Theorem 5.8. The functor G_s is called (Joseph's version of) *completion functor* and was defined in the above form in [Jo2]. The definition was inspired by another (Enright's version of) completion functor, defined earlier in [En] in a more restrictive situation. The name *completion functor* is motivated by the property, dual to Proposition 6.6(a), which can be understood in the way that all Verma modules can be obtained from the antidominant Verma module by a certain completion process with respect to the action of various $\mathfrak{sl}_2$-subalgebras. Connection between completion functors and twisting functors is clarified by the following:

Theorem 6.11 ([KhMa2])**.** *The functor G_s is right adjoint to the functor T_s.*

In particular, it follows that T_s corresponds, under the equivalence of Theorem 5.8, to C_s^r. From the previous subsection we obtain that $\mathrm{G}_s \cong \star \circ \mathrm{T}_s \circ \star$ and that completion functors satisfy braid relations. This allows us to define the completion functor G_w for every $w \in W$. The action of derived completion functors on the bounded derived category of $\mathcal{O}_\lambda$ gives a naïve categorification of the left regular $\mathbb{Z}[\mathbb{S}_n]$-module.

Example 6.12. The action of G_s on the principal block of the category $\mathcal{O}$ for the algebra $\mathfrak{gl}_2$ is given in the following table:

M	e	s	e s	s e	s e s
$\mathrm{G}_s\, M$	0	e s	e s	s	s e s

6.4. **Alternative description via (co)approximations.** Let A be a finite dimensional associative $\Bbbk$-algebra and $e \in A$ an idempotent. Then we have the following natural pair of adjoint functors:

$$A\text{-mod} \underset{Ae\otimes_{eAe}{}_-}{\overset{\mathrm{Hom}_A(Ae,{}_-)}{\rightleftarrows}} eAe\text{-mod} \tag{6.1}$$

Denote by $\mathcal{X}_e$ the full subcategory of A-mod, which consists of all modules M admitting a presentation $X_1 \to X_0 \twoheadrightarrow M$ with $X_0, X_1 \in \mathrm{add}(Ae)$. Then (6.1) restricts to the following equivalence of categories, see [Au, Section 5]:

$$\mathcal{X}_e \underset{Ae\otimes_{eAe}{}_-}{\overset{\mathrm{Hom}_A(Ae,{}_-)}{\rightleftarrows}} eAe\text{-mod}.$$

As mentioned in Subsection 1.6, the category $\mathcal{X}_e$ is equivalent to the quotient of A-mod by the Serre subcategory consisting of all modules M such that $eM = 0$. Set

$$\mathrm{P}_e := Ae \otimes_{eAe} \mathrm{Hom}_A(Ae, {}_-) : A\text{-mod} \to A\text{-mod}.$$

This functor is right exact (since $\mathrm{Hom}_A(Ae, {}_-)$ is exact because of the projectivity of Ae) and hence is isomorphic to the functor $Ae \otimes_{eAe} eA \otimes_A {}_-$. We also have the right exact functor $\mathrm{R}_e := AeA \otimes_A {}_-$. The functor P_e is called *coapproximation* with respect to Ae and the functor R_e is called *partial coapproximation* with respect to Ae. We also have the corresponding right adjoint functors $\overline{\mathrm{P}}_e$ and $\overline{\mathrm{R}}_e$ of approximation and partial approximation with respect to the injective module $\mathrm{Hom}_A(eA, \Bbbk)$, respectively.

The functor R_e can be understood as the composition of the following two steps (see [KhMa2]):

- Given an A-module M consider some projective cover $P_M \twoheadrightarrow M$ of M, and denote by M' the quotient of P_M modulo the trace of Ae in the kernel of this projective cover.
- Consider the trace M'' of Ae in M'. We have $M'' \cong \mathrm{R}_e\, M$.

Though none of these two steps is functorial, their composition turns out to be functorial, in particular, independent of the choice of P_M. The functor $\overline{\mathrm{R}}_e$ can be described dually using injective modules.

Let now λ be integral and dominant. For $\mu \in W \cdot \lambda$ let e_μ denote the primitive idempotent of B_λ corresponding to $P(\mu)$. For a simple reflection s let e_s denote the sum of all e_μ such that $\mu \geq s \cdot \mu$.

Theorem 6.13 ([KhMa2, MS2]). *Let s be a simple reflection. Then we have:*

$$\mathrm{T}_s \cong \mathrm{R}_{e_s}, \quad \mathrm{T}_s^2 \cong \mathrm{P}_{e_s}, \quad \mathrm{G}_s \cong \overline{\mathrm{R}}_{e_s}, \quad \mathrm{G}_s^2 \cong \overline{\mathrm{P}}_{e_s}.$$

Theorem 6.13 can be extended to the following situation: let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra and $W_{\mathfrak{p}}$ the corresponding parabolic subgroup of W. Denote by $e_{\mathfrak{p}}$ the sum of all e_μ such that $\mu \geq s \cdot \mu$ for any simple reflection $s \in W_{\mathfrak{p}}$. Let $w_o^{\mathfrak{p}}$ denote the longest element of $W_{\mathfrak{p}}$.

Theorem 6.14 ([KhMa2, MS2]). *We have:*

$$\mathrm{T}_{w_o^{\mathfrak{p}}} \cong \mathrm{R}_{e_{\mathfrak{p}}}, \quad \mathrm{T}^2_{w_o^{\mathfrak{p}}} \cong \mathrm{P}_{e_{\mathfrak{p}}}, \quad \mathrm{G}_{w_o^{\mathfrak{p}}} \cong \overline{\mathrm{R}}_{e_{\mathfrak{p}}}, \quad \mathrm{G}^2_{w_o^{\mathfrak{p}}} \cong \overline{\mathrm{P}}_{e_{\mathfrak{p}}}.$$

6.5. **Serre functor.** Let $\mathscr{C}$ be a $\Bbbk$-linear additive category with finite dimensional morphism spaces. A *Serre functor* on $\mathscr{C}$ is an additive auto-equivalence F of $\mathscr{C}$ together with isomorphisms

$$\Psi_{x,y} : \mathscr{C}(x, \mathrm{F}\, y) \cong \mathscr{C}(y, x)^*,$$

for all $x, y \in \mathscr{C}$, natural in x and y (see [BoKa]). If a Serre functor exists, it is unique (up to isomorphism) and commutes with all auto-equivalences of $\mathscr{C}$.

Proposition 6.15 ([Ha]). *If A is a finite dimensional $\Bbbk$-algebra of finite global dimension, then the left derived of the* Nakayama functor

$$\mathrm{N} := \mathrm{Hom}_A(\_, A)^* \cong A^* \otimes_A \_$$

is a Serre functor on $\mathcal{D}^b(A)$.

Proposition 6.15 implies existence of a Serre functor on $\mathcal{D}^b(\mathcal{O}_\lambda)$. It turns out that this functor can be described in terms of both shuffling and twisting functors.

Theorem 6.16 ([MS3]). *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular. Then the Nakayama functor on $\mathcal{O}_\lambda$ is isomorphic to both $\mathrm{T}^2_{w_o}$ and $\mathrm{C}^2_{w_o}$, in particular, the latter two functors are isomorphic.*

The Serre functor on $\mathcal{O}_\lambda$ is described geometrically in [BBM]. A description of the Serre functor in terms of Harish-Chandra bimodules can be found in [MM1]. As we have seen, the braid group acts on $\mathcal{D}^b(\mathcal{O}_\lambda)$ both via derived twisting and shuffling functors. The Serre functor commutes with all auto-equivalences, hence with all functors from these actions. Note that the Serre functor is given by $\mathrm{T}^2_{w_o}$ (or $\mathrm{C}^2_{w_o}$) and the element w_o^2 generates the center of the braid group.

7. Category $\mathcal{O}$: grading and combinatorics

7.1. **Double centralizer property.** We start with the following observation:

Proposition 7.1. *For any integral $\mu \in \mathfrak{h}^*$ the module $P(\mu)$ is injective, in particular, tilting, if and only if μ is antidominant.*

Idea of the proof. The integral weight $-\rho$ satisfies $W_{-\rho} = W$. Hence the block $\mathcal{O}_{-\rho}$ contains exactly one simple module $L(-\rho)$, which is a dominant Verma module, thus projective. This implies that $\mathcal{O}_{-\rho}$ is semi-simple and hence $L(-\rho)$ is also injective. From Proposition 5.3(f) it follows that the module $\theta\,\Delta(-\rho)$ is both projective and injective for any projective functor θ, which shows that $P(\mu)$ is injective for all antidominant μ.

On the other hand, any projective module has a Verma flag and every Verma has simple socle, which is given by an antidominant weight. □

Now let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and dominant. Then, by the BGG reciprocity, we have $\Delta(\lambda) \hookrightarrow P(w_o \cdot \lambda)$ and, since $P(w_o \cdot \lambda)$ is a tilting module, the cokernel of this embedding has a Verma flag. In particular, the injective envelope of this cokernel is in $\mathrm{add}(P(w_o \cdot \lambda))$. This gives us an exact sequence of the form

$$0 \to \Delta(\lambda) \to P(w_o \cdot \lambda) \to X$$

with $X \in \mathrm{add}(P(w_o \cdot \lambda))$. Applying θ_w, $w \in W$, and summing up over all w we obtain the exact sequence

$$0 \to P_\lambda \to X_0 \to X_1, \tag{7.1}$$

with $X_0, X_1 \in \mathrm{add}(P(w_o \cdot \lambda))$. Denote by C_λ the endomorphism algebra of $P(w_o \cdot \lambda)$. Using the standard results on quasi-Frobenius ring (see [Ta, KSX]), we obtain the following theorem, known as Soergel's *Struktursatz*:

Theorem 7.2 ([So1])**.** *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and dominant.*

(*a*) *The functor*

$$\mathbb{V} = \mathbb{V}_\lambda := \mathrm{Hom}_{\mathfrak{g}}(P(w_o \cdot \lambda), {}_-) : \mathcal{O}_\lambda \to C_\lambda^{\mathrm{op}}\text{-mod}$$

is full and faithful on projective modules.

(*b*) *The B_λ-module $\mathrm{Hom}_{\mathfrak{g}}(P_\lambda, P(w_o \cdot \lambda))$ has the* double centralizer property*, that is the action of B_λ on $\mathrm{Hom}_{\mathfrak{g}}(P_\lambda, P(w_o \cdot \lambda))$ coincides with the centralizer of the action of the endomorphism algebra of this module.*

The functor $\mathbb{V}$ is called *Soergel's combinatorial* functor.

7.2. **Endomorphisms of the antidominant projective.** Theorem 7.2 reduces projective modules in the category $\mathcal{O}_\lambda$ to certain (not projective) modules over the endomorphism algebra C_λ of the unique indecomposable projective-injective module $P(w_o \cdot \lambda)$ in $\mathcal{O}_\lambda$. It is very natural to ask what the algebra C_λ is. The answer is given by the following theorem. Consider the polynomial algebra $\mathbb{C}[\mathfrak{h}]$. Recall that $W \cong \mathbb{S}_n$ acts on $\mathfrak{h}$, which also induces an action of W on $\mathbb{C}[\mathfrak{h}]$. We regard $\mathbb{C}[\mathfrak{h}]$ as a (positively) *graded* algebra by putting $\mathfrak{h}$ in degree 2. We have the (graded) subalgebra $\mathbb{C}[\mathfrak{h}]^W$ of W-invariants in $\mathbb{C}[\mathfrak{h}]$. Denote by $(\mathbb{C}[\mathfrak{h}]^W_+)$ the ideal of $\mathbb{C}[\mathfrak{h}]$ generated by all homogeneous invariants of positive degree. Then we have the corresponding quotient $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+)$, called the *coinvariant* algebra. The coinvariant algebra is a finite dimensional associative and commutative graded algebra of dimension $|W|$. The "strange" grading on $\mathbb{C}[\mathfrak{h}]$ (i.e. $\mathfrak{h}$ being of degree two) is motivated by the realization of the coinvariant algebra as the cohomology algebra of a flag variety, see [Hi]. If λ is singular and W_λ is the dot-stabilizer of λ, then we have the (graded) subalgebra $(\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+))^{W_\lambda}$ of W_λ-invariants in $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+)$. We have the following theorem, know as Soergel's *Endomorphismensatz*:

Theorem 7.3 ([So1])**.** *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and dominant.*

(*a*) *The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ surjects onto C_λ, in particular, C_λ is commutative.*

(*b*) *The algebras C_λ and $(\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+))^{W_\lambda}$ are isomorphic.*

Corollary 7.4. *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and dominant and Id_λ the identity functor on $\mathcal{O}_\lambda$. Then the evaluation map*

$$\begin{array}{rccc} \mathrm{ev}: & \mathrm{End}(\mathrm{Id}_\lambda) & \to & \mathrm{End}_{\mathfrak{g}}(P(w_o \cdot \lambda)) \\ & \varphi & \mapsto & \varphi_{P(w_o\cdot\lambda)} \end{array}$$

is an isomorphism of algebras. In particular, the center of B_λ is isomorphic to C_λ.

7.3. **Grading on B_λ.** The functor $\mathbb{V}$ induces an equivalence between the additive category of projective modules in $\mathcal{O}_\lambda$ and the additive subcategory $\mathrm{add}(\mathbb{V}P_\lambda)$ of C_λ-mod. All projective functors are endofunctors of the category of projective modules in $\mathcal{O}_\lambda$. Hence it is natural to ask what are the images of projective functors under $\mathbb{V}$. It turns out that these images have a very nice description. Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular and $\mu \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and such that $W_\mu = \{e, s\}$ for some simple reflection $s \in S$. Then C_λ is the whole coinvariant algebra for W and C_μ is the subalgebra of s-invariants in C_λ. Recall from (5.1) that the projective endofunctor θ_s of $\mathcal{O}_\lambda$ is the composition of $\theta_{\lambda,\mu}$ and $\theta_{\mu,s\cdot\lambda}$

Theorem 7.5 ([So1]). *There are isomorphisms of functors as follows:*
(*a*) $\mathbb{V}_\mu \circ \theta_{\lambda,\mu} \cong \mathrm{Res}^{C_\lambda}_{C_\mu} \circ \mathbb{V}_\lambda$;
(*b*) $\mathbb{V}_\lambda \circ \theta_{\mu,s\cdot\lambda} \cong \mathrm{Ind}^{C_\lambda}_{C_\mu} \circ \mathbb{V}_\mu$;
(*c*) $\mathbb{V} \circ \theta_s \cong C_\lambda \otimes_{C_\mu} \mathbb{V}_{-}$.

Consider the category C_λ-gmod of *finite dimensional graded* C_λ-modules (here morphisms are homogeneous maps of degree zero). Note that $\mathbb{V}P(\lambda) \cong \mathbb{C}$ can be considered as a graded C_λ-module concentrated in degree zero. The subalgebra C_μ of C_λ is graded and C_λ is free as a C_μ-module with generators being of degrees zero and two. It follows that the functor $C_\lambda \otimes_{C_\mu} {}_-$ is an exact endofunctor of C_λ-gmod. Since every indecomposable projective in $\mathcal{O}_\lambda$ is a direct summand of some module of the form $\theta_{s_1}\theta_{s_2}\cdots\theta_{s_k} P(\lambda)$, by standard arguments (see e.g. [St1]) we obtain that every $\mathbb{V}P(w\cdot\lambda)$, $w \in W$, can be considered as an object in C_λ-gmod (unique up to isomorphism and shift of grading).

We fix a grading of $\mathbb{V}P(w\cdot\lambda)$ such that its unique simple quotient is concentrated in degree zero. This fixes a grading on the module $\mathbb{V}P_\lambda$ and thus induces a grading on its endomorphism algebra B_λ. We denote by $\mathcal{O}^{\mathbb{Z}}_\lambda := B_\lambda$-gmod the category of *finite dimensional graded* B_λ-modules and consider this category as a graded version of the category $\mathcal{O}_\lambda$. Note that not all modules in $\mathcal{O}_\lambda$ are gradable, see [St2].

Theorem 7.5(a) provides a graded interpretation of translation to the wall. This induces a canonical grading on B_λ for any integral $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$. For any integral $\lambda, \mu \in \mathfrak{h}^*_{\mathrm{dom}}$ and any indecomposable projective functor $\theta_{\lambda,\nu} : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ there is a unique (up to isomorphism) *graded lift* of $\theta_{\lambda,\nu}$, that is a functor from B_λ-gmod $\to$ B_μ-gmod which maps the graded module $P(\lambda)$ to the graded module $P(\nu)$ (note that there are no graded shifts here!) and is isomorphic to $\theta_{\lambda,\nu}$ after forgetting the grading. Abusing notation, we denoted this graded lift of $\theta_{\lambda,\nu}$ also by $\theta_{\lambda,\nu}$. All this was worked out in [St2] in details.

Example 7.6. Let $n = 2$. Then the algebra B_0 is given by (4.1) and it is easy to check that the above procedure results in the following grading:

degree i	component $(B_0)_i$
0	$\mathtt{1}_e, \mathtt{1}_s$
1	a, b
2	ba

One observes that this grading is *positive*, that is all nonzero components have non-negative degrees, and the zero component is semisimple.

Abusing notation we will denote *standard graded lifts* in $\mathcal{O}^{\mathbb{Z}}_\lambda$ of structural modules from $\mathcal{O}_\lambda$ in the same way. Thus $P(\lambda)$ is the standard graded lift described above. It has simple top $L(\lambda)$ concentrated in degree zero. Further, $\Delta(\lambda)$ is a graded quotient of $P(\lambda)$. The duality $\star$ lifts to $\mathcal{O}^{\mathbb{Z}}_\lambda$ in the standard way such that for $M \in \mathcal{O}^{\mathbb{Z}}_\lambda$ we have $(M^\star)_i = (M_{-i})^*$, $i \in \mathbb{Z}$. Using the graded version of $\star$ we obtain standard graded lifts of $I(\lambda)$ and $\nabla(\lambda)$. The standard graded lift of $T(\lambda)$ is defined so that the unique up to scalar inclusion $\Delta(\lambda) \hookrightarrow T(\lambda)$ is homogeneous of degree zero. Note that all canonical maps between structural modules are homogeneous of degree zero.

Example 7.7. Let $n = 2$. Here are graded filtration of all standard graded lifts of structural modules in B_0-gmod (as usual, we abbreviate $L(e) := L(0)$ by e and $L(s) := L(s \cdot 0)$ by s).

degree	$L(e)$	$L(s) = T(s) = \Delta(s) = \nabla(s)$	$\Delta(e) = P(e)$	$P(s)$	$I(e) = \nabla(e)$	$I(s)$	$T(e)$
-2						s	
-1					s	e	s
0	e	s	e	s	e	s	e
1			s	e			s
2				s			

After Example 7.6 it is natural to ask whether the grading on B_λ will always be positive. It turns out that the answer is "yes", but to motivate and explain it we would need to "upgrade" the Weyl group to the Hecke algebra.

7.4. **Hecke algebra.** Denote by $\mathbb{H} = \mathbb{H}_n = \mathbb{H}(W, S)$ the *Hecke algebra* of W. It is defined as a free $\mathbb{Z}[v, v^{-1}]$-module with the *standard* basis $\{H_x : x \in W\}$ and multiplication given by

$$H_x H_y = H_{xy} \text{ if } \mathfrak{l}(x) + \mathfrak{l}(y) = \mathfrak{l}(xy), \text{ and } H_s^2 = H_e + (v^{-1} - v)H_s \text{ for } s \in S. \tag{7.2}$$

The algebra $\mathbb{H}$ is a deformation of the group algebra $\mathbb{Z}[W]$. As a $\mathbb{Z}[v, v^{-1}]$-algebra, it is generated by $\{H_s : s \in S\}$, or (which will turn out to be more convenient) by the set $\{\underline{H}_s = H_s + vH_e : s \in S\}$.

There is a unique involution $^-$ on $\mathbb{H}$ which maps $v \mapsto v^{-1}$ and $H_s \mapsto (H_s)^{-1}$. Note that this involution fixes all $\underline{H}_s$. More general, $\mathbb{H}$ has a unique basis, called *Kazhdan-Lusztig basis*, which consists of fixed under $^-$ elements $\underline{H}_x$, $x \in W$, such that $\underline{H}_x = H_x + \sum_{y \in W,\, y \neq x} h_{y,x} H_y$, where $h_{y,x} \in v\mathbb{Z}[v]$ (here we use the normalization of [So4]). Set $h_{x,x} := 1$. The polynomials $h_{y,x}$ are called *Kazhdan-Lusztig polynomials* and were defined in [KaLu].

With respect to the generators $\underline{H}_s$, $s \in S$, we have the following set of defining relations for $\mathbb{H}$:

$$\begin{array}{rcll} \underline{H}_s^2 & = & (v + v^{-1})\underline{H}_s; & \\ \underline{H}_s \underline{H}_t & = & \underline{H}_t \underline{H}_s, & \text{if } ts = st; \\ \underline{H}_s \underline{H}_t \underline{H}_s + \underline{H}_t & = & \underline{H}_t \underline{H}_s \underline{H}_t + \underline{H}_s, & \text{if } tst = sts. \end{array}$$

The algebra $\mathbb{H}$ has a symmetrizing trace form $\tau : \mathbb{H} \to \mathbb{Z}[v, v^{-1}]$, which sends H_e to 1 and H_w to 0 for $w \neq e$. With respect to τ we have the *dual Kazhdan-Lusztig basis* $\{\hat{\underline{H}}_x : x \in W\}$, defined via $\tau(\hat{\underline{H}}_x \underline{H}_{y^{-1}}) = \delta_{x,y}$.

Example 7.8. In the case $n = 3$ we have two simple reflections s and t, $W = \{e, s, t, st, ts, sts = tst\}$ and the elements of the Kazhdan-Lusztig basis are given by the following table:

$$\begin{array}{rclrcl} \underline{H}_e & = & H_e, & \underline{H}_{st} & = & H_{st} + vH_s + vH_t + v^2 H_e, \\ \underline{H}_s & = & H_s + vH_e, & \underline{H}_{ts} & = & H_{ts} + vH_s + vH_t + v^2 H_e, \\ \underline{H}_t & = & H_t + vH_e, & \underline{H}_{sts} & = & H_{sts} + vH_{st} + vH_{ts} + v^2 H_s + v^2 H_t + v^3 H_e. \end{array}$$

The elements of the dual Kazhdan-Lusztig basis are given by the following table:

$$\begin{array}{rclrcl} \hat{\underline{H}}_e & = & H_e - vH_s - vH_t + v^2 H_{st} + v^2 H_{ts} - v^3 H_{sts}, & \hat{\underline{H}}_{st} & = & H_{st} - vH_{sts}, \\ \hat{\underline{H}}_s & = & H_s - vH_{st} - vH_{ts} + v^2 H_{sts}, & \hat{\underline{H}}_{ts} & = & H_{ts} - vH_{sts}, \\ \hat{\underline{H}}_t & = & H_t - vH_{st} - vH_{ts} + v^2 H_{sts}, & \hat{\underline{H}}_{sts} & = & H_{sts}. \end{array}$$

Let $\mathbb{F}$ be any commutative ring and $\iota : \mathbb{Z}[v, v^{-1}] \to \mathbb{F}$ be a homomorphism of unitary rings. Then we have the *specialized* Hecke algebra $\mathbb{H}^{(\mathbb{F},\iota)} = \mathbb{F} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{H}$. Again, if ι is clear from the context (for instance if ι is the natural inclusion), we will omit it in the notation.

7.5. **Categorification of the right regular $\mathbb{H}$-module.** For any regular and integral $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ we have the following:

Lemma 7.9. *Let $s \in S$ be a simple reflection.*

(*a*) *For any $w \in W$ with $ws > w$ there is an exact sequence in $\mathcal{O}^{\mathbb{Z}}_\lambda$ as follows:*
$$0 \to \Delta(w \cdot \lambda)\langle -1 \rangle \to \theta_s \Delta(w \cdot \lambda) \to \Delta(ws \cdot \lambda) \to 0.$$
(*b*) *For any $w \in W$ with $ws < w$ there is an exact sequence in $\mathcal{O}^{\mathbb{Z}}_\lambda$ as follows:*
$$0 \to \Delta(ws \cdot \lambda) \to \theta_s \Delta(w \cdot \lambda) \to \Delta(w \cdot \lambda)\langle 1 \rangle \to 0.$$

Lemma 7.9 implies the following "upgrade" of Proposition 5.7 to $\mathcal{O}^{\mathbb{Z}}_\lambda$:

Proposition 7.10 (Categorification of the right regular $\mathbb{H}$-module)**.** *Let λ be dominant, regular and integral.*

(*a*) *There is a unique isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules $\varphi : \mathbb{H} \to [\mathcal{O}^{\mathbb{Z}}_\lambda]$ such that $\varphi(H_w) = [\Delta(w \cdot \lambda)]$ for all $w \in W$.*

(*b*) *For any $s \in S$ the following diagram commutes:*
$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\cdot \underline{H}_s} & \mathbb{H} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ [\mathcal{O}^{\mathbb{Z}}_\lambda] & \xrightarrow{[\theta_s]\cdot} & [\mathcal{O}^{\mathbb{Z}}_\lambda] \end{array}$$

However, now we can say much more, namely:

Theorem 7.11 (Graded reformulation of the Kazhdan-Lusztig conjecture)**.** *For any $w \in W$ the following diagram commutes:*
$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\cdot \underline{H}_w} & \mathbb{H} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ [\mathcal{O}^{\mathbb{Z}}_\lambda] & \xrightarrow{[\theta_w]\cdot} & [\mathcal{O}^{\mathbb{Z}}_\lambda] \end{array}$$

Kazhdan-Lusztig conjecture was formulated in a different (equivalent) way in [KaLu] and proved in [BB, BrKa]. It has many important combinatorial consequences. The most basic ones are described in the next subsection.

7.6. **Combinatorics of $\mathcal{O}_\lambda$.** Recall first that for $x \in W$ the indecomposable projective functor θ_x was uniquely defined by the property $\theta_x\, \Delta(\lambda) = P(x \cdot \lambda)$. As every projective in $\mathcal{O}_\lambda$ has a standard filtration, going to the Grothendieck group, Theorem 7.11 implies that the (graded) multiplicities of standard modules in a standard filtration of an indecomposable projective module in $\mathcal{O}_\lambda$ are given by Kazhdan-Lusztig polynomials as follows:

Corollary 7.12. *For $x \in W$ we have*
$$[P(x \cdot \lambda)] = \varphi(\underline{H}_x) = \sum_{y \in W} h_{y,x} [\Delta(y \cdot \lambda)].$$

Using the BGG-reciprocity we obtain that the (graded) composition multiplicities of Verma modules are also given by Kazhdan-Lusztig polynomials as follows:

Corollary 7.13. *For $y \in W$ we have*

$$[\Delta(y \cdot \lambda)] = \sum_{x \in W} h_{y,x}[L(x \cdot \lambda)].$$

We would like to categorify (i.e. present a categorical analogue for) the form τ. Define the linear map $\Phi : [\mathcal{O}_\lambda^{\mathbb{Z}}] \to \mathbb{Z}[v, v^{-1}]$ as follows. For $M \in \mathcal{O}_\lambda^{\mathbb{Z}}$ set

$$\Phi([M]) := \sum_{i \in \mathbb{Z}} \dim \mathrm{Hom}(\Delta(\lambda)\langle i \rangle, M) v^{-i}.$$

This is well-defined as $\Delta(\lambda)$ is projective. Then, for any $M \in \mathcal{O}_\lambda^{\mathbb{Z}}$ we have $\Phi([M]) = \tau(\varphi^{-1}([M]))$ (this is enough to check for $M = \Delta(x \cdot \lambda)$, $x \in W$, in which case it is clear). The form Φ implies the following:

Corollary 7.14. *For $y \in W$ we have $[L(y \cdot \lambda)] = \varphi(\underline{\hat{H}}_y)$.*

Idea of the proof. For $x, y \in W$, using adjunction and definitions, we have

$$\begin{aligned}
\Phi([\theta_x L(y^{-1} \cdot \lambda)]) &= \sum_{i \in \mathbb{Z}} \dim \mathrm{Hom}(\Delta(\lambda)\langle i \rangle, \theta_x L(y^{-1} \cdot \lambda)) v^{-i} \\
&= \sum_{i \in \mathbb{Z}} \dim \mathrm{Hom}(\theta_{x^{-1}} \Delta(\lambda)\langle i \rangle, L(y^{-1} \cdot \lambda)) v^{-i} \\
&= \sum_{i \in \mathbb{Z}} \dim \mathrm{Hom}(P(x^{-1} \cdot \lambda)\langle i \rangle, L(y^{-1} \cdot \lambda)) v^{-i} \\
&= \delta_{x,y}
\end{aligned}$$

and the claim follows from Corollary 7.12. □

Corollary 7.14 says that, categorically, the dual Kazhdan-Lusztig basis is the "most natural" basis of $\mathbb{H}$ as it corresponds to the "most natural" basis of the Grothendieck group of $\mathcal{O}_\lambda^{\mathbb{Z}}$ consisting of the classes of simple modules. Later on we will see that all "nice" categorifications of $\mathbb{H}$-modules have a dual Kazhdan-Lusztig basis. From the above we have that all composition subquotients of all standard lifts of indecomposable projective modules in $\mathcal{O}_\lambda^{\mathbb{Z}}$ live in non-negative degrees with only simple top being in degree zero. Hence we get:

Corollary 7.15. *The algebra B_λ is positively graded.*

8. $\mathbb{S}_n$-categorification: Soergel bimodules, cells and Specht modules

8.1. **Soergel bimodules.** Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular. Then we have the (strict) monoidal category $\mathscr{S}_\lambda$ of projective endofunctors on $\mathcal{O}_\lambda$. As was mentioned in the previous section, both $\mathcal{O}_\lambda$ and $\mathscr{S}_\lambda$ admit graded lifts, which we can formalize as follows. Denote by $\mathscr{S} = \mathscr{S}_n$ the 2-category with unique object $\mathtt{i}$ and $\mathscr{S}(\mathtt{i}, \mathtt{i}) = \mathscr{S}_\lambda$ (see [Bac2] for a description of homomorphisms between projective functors). Denote also by $\mathscr{S}^{\mathbb{Z}}$ the 2-category with unique object $\mathtt{i}$, whose 1-morphisms are all endofunctors of $\mathcal{O}_\lambda^{\mathbb{Z}}$ isomorphic to finite direct sums of graded shifts of standard graded lifts of θ_w, $w \in W$; and 2-morphisms are all natural transformations of functors which are homogeneous of degree zero. The category $\mathscr{S}$ is a fiat category and the category $\mathscr{S}^{\mathbb{Z}}$ is a $\mathbb{Z}$-cover of $\mathscr{S}$ (in the sense that there is a free action of $\mathbb{Z}$ on $\mathscr{S}^{\mathbb{Z}}$, by grading shifts, such that the quotient is isomorphic to $\mathscr{S}$ and thus endows $\mathscr{S}$ with the structure of a graded category).

Using Soergel's combinatorial description of $\mathcal{O}_\lambda$, indecomposable 1-morphisms of $\mathscr{S}$ and $\mathscr{S}^{\mathbb{Z}}$ can be described, up to isomorphism and graded shift, in the following way: Let $w \in W$ and $w = s_1 s_2 \cdots s_k$ be a reduced decomposition of w. Consider the graded C_λ-C_λ-bimodule

$$D_w := C_\lambda \otimes_{C_\lambda^{s_k}} C_\lambda \otimes_{C_\lambda^{s_{k-1}}} \cdots \otimes_{C_\lambda^{s_2}} C_\lambda \otimes_{C_\lambda^{s_1}} C_\lambda \langle \mathfrak{l}(w) \rangle.$$

Define the C_λ-C_λ-bimodules $\underline{D}_w$ recursively as follows: $\underline{D}_e = D_e = C_\lambda$; for $\mathfrak{l}(w) > 0$ let $\underline{D}_w$ be the unique indecomposable direct summand of D_w which is not isomorphic (up to shift of grading) to $\underline{D}_x$ for some x such that $\mathfrak{l}(x) < \mathfrak{l}(w)$. From Subsection 7.3 we derive that the bimodule $\underline{D}_w$ realizes the action of θ_w on the level of C_λ-mod.

Proposition 8.1. *For every $w \in W$ there is an isomorphism of graded functors as follows:* $\mathbb{V} \circ \theta_w(\_) \cong \underline{D}_w \otimes_{C_\lambda} \circ \mathbb{V}(\_)$.

Bimodules $\underline{D}_w$, $w \in W$, are called *Soergel bimodules* and were introduced in [So1, So2]. Usually they are defined in the deformed version as bimodules over the polynomial ring $\mathbb{C}[\mathfrak{h}]$. The disadvantage of the latter definition (which corresponds to the action of projective functors on the category ${}^{\infty}_{0}\mathcal{H}^{\infty}_{0}$) is that it does not produce a fiat category. Therefore in this paper we will restrict ourselves to the case of Soergel bimodules over the coinvariant algebra. Soergel's combinatorial functor $\mathbb{V}$ provides a biequivalence between the 2-category $\mathscr{S}^{\mathbb{Z}}$ and the 2-category with one object whose endomorphism category is the minimal full fully additive subcategory of the category of C_λ-C_λ-bimodules containing all Soergel bimodules and closed under isomorphism and shifts of grading.

Theorem 8.2 (Categorification of the Hecke algebra)**.** *The map*

$$\begin{array}{ccc} [\mathscr{S}^{\mathbb{Z}}(\mathtt{i},\mathtt{i})] & \longrightarrow & \mathbb{H} \\ [\theta_w] & \mapsto & \underline{H}_w \end{array}$$

induces an anti-isomorphism of unital $\mathbb{Z}[v, v^{-1}]$-rings.

Now we would like to study 2-representation of the 2-categories $\mathscr{S}$ and $\mathscr{S}^{\mathbb{Z}}$. Note that both categories come together with the canonical *natural* 2-representation, namely the action of $\mathscr{S}$ on $\mathcal{O}_\lambda$ and of $\mathscr{S}^{\mathbb{Z}}$ on $\mathcal{O}^{\mathbb{Z}}_\lambda$. The idea now is to use these natural representations to construct other 2-representation.

8.2. **Kazhdan-Lusztig cells.** The 2-category $\mathscr{S}$ is a fiat category and hence we have the corresponding notions of left, right and two-sided cells as defined in Subsection 3.4. Indecomposable objects of $\mathscr{S}$ correspond to elements of the Kazhdan-Lusztig basis in $\mathbb{H}$. To be able to describe cells for $\mathscr{S}$ we need to know structure constants of $\mathbb{H}$ with respect to the Kazhdan-Lusztig basis.

For different $x, y \in W$ denote by $\mu(y, x)$ the coefficient of v in the Kazhdan-Lusztig polynomial $h_{y,x}$ from Subsection 7.4. The function μ is called *Kazhdan-Lusztig μ-function.* Its importance is motivated by the following (see [So4] for our normalization):

Proposition 8.3 ([KaLu])**.** *For any $x \in W$ and any simple reflection s we have*

$$\underline{H}_x \underline{H}_s = \begin{cases} \underline{H}_{xs} + \sum_{y<x, ys<y} \mu(y,x) \underline{H}_y, & xs > x; \\ (v+v^{-1})\underline{H}_x, & xs < x; \end{cases}$$

and

$$\underline{H}_s \underline{H}_x = \begin{cases} \underline{H}_{sx} + \sum_{y<x, sy<y} \mu(y,x) \underline{H}_y, & sx > x; \\ (v+v^{-1})\underline{H}_x, & sx < x. \end{cases}$$

Example 8.4. Here is the essential part of the multiplication table for $\mathbb{H}$ in the Kazhdan-Lusztig basis for $n = 3$ (see Example 7.8):

$*$	$\underline{H}_e$	$\underline{H}_s$	$\underline{H}_t$	$\underline{H}_{st}$	$\underline{H}_{ts}$	$\underline{H}_{sts}$
$\underline{H}_e$	$\underline{H}_e$	$\underline{H}_s$	$\underline{H}_t$	$\underline{H}_{st}$	$\underline{H}_{ts}$	$\underline{H}_{sts}$
$\underline{H}_s$	$\underline{H}_s$	$(v+v^{-1})\underline{H}_s$	$\underline{H}_{st}$	$(v+v^{-1})\underline{H}_{st}$	$\underline{H}_{sts}+\underline{H}_s$	$(v+v^{-1})\underline{H}_{sts}$
$\underline{H}_t$	$\underline{H}_t$	$\underline{H}_{ts}$	$(v+v^{-1})\underline{H}_t$	$\underline{H}_{sts}+\underline{H}_t$	$(v+v^{-1})\underline{H}_{ts}$	$(v+v^{-1})\underline{H}_{sts}$

For $x, y \in W$ write $x \leq_L y$ provided that there is $z \in W$ such that $\underline{H}_y$ occurs with a nonzero coefficient in the decomposition of $\underline{H}_z \underline{H}_x$ in the Kazhdan-Lusztig basis. Then $\leq_L$ is a partial pre-order on W. Define $\leq_R$ and $\leq_{LR}$ similarly for the right and the two-sided multiplications, respectively. Equivalence classes with respect to $\leq_L$, $\leq_R$ and $\leq_{LR}$ are called *Kazhdan-Lusztig left, right and two-sided cells*, respectively. We denote the corresponding equivalence relations by $\sim_L$, $\sim_R$ and $\sim_{LR}$, respectively. It turns out that the latter can be given a nice combinatorial description. Recall that the Robinson-Schensted correspondence associates to every $w \in W$ a pair $(p(w), q(w))$ of standard Young tableaux of the same shape, see [Sa, Section 3.1].

Proposition 8.5 ([KhLa]). *For $x, y \in W$ we have the following:*

(*a*) *$x \sim_R y$ if and only if $p(x) = p(y)$.*
(*b*) *$x \sim_L y$ if and only if $q(x) = q(y)$.*
(*c*) *$x \sim_{LR} y$ if and only if $p(x)$ and $p(y)$ have the same shape.*

Example 8.6. In the case $n = 3$ we have the following

$$\begin{array}{rllll} \text{right cells:} & \{e\}, & \{s, st\}, & \{t, ts\}, & \{sts\}; \\ \text{left cells:} & \{e\}, & \{s, ts\}, & \{t, st\}, & \{sts\}; \\ \text{two-sided cells:} & \{e\}, & \{s, st, t, ts\}, & \{sts\}. & \end{array}$$

8.3. **Cell modules.** Fix now a left cell $\mathcal{L}$ of W. Then the $\mathbb{Z}[v, v^{-1}]$-linear span $\mathbb{Z}[v, v^{-1}]\mathcal{L}$ of $\underline{H}_w$, $w \in \mathcal{L}$, has the natural structure of an $\mathbb{H}$-module, given by the left multiplication with the elements in the Kazhdan-Lusztig basis (and treating all vectors which do not belong to $\mathbb{Z}[v, v^{-1}]\mathcal{L}$ as zero). This module is called the *cell module* corresponding to $\mathcal{L}$. Similarly one defines a right cell $\mathbb{H}$-module for every right cell $\mathcal{R}$. Note that the cell module $\mathbb{Z}[v, v^{-1}]\mathcal{L}$ comes along with a distinguished $\mathbb{Z}[v, v^{-1}]$-bases, namely the Kazhdan-Lusztig basis $\{\underline{H}_w : w \in \mathcal{L}\}$.

Proposition 8.7 ([KaLu, Na]). *Let $\mathcal{L} \subset W$ be a left cell and $\lambda \vdash n$ such that $p(w)$ has shape λ for all $w \in \mathcal{L}$. Then the W-module obtained by complexification and generic specialization of $\mathbb{Z}[v, v^{-1}]\mathcal{L}$ is isomorphic to the Specht W-module corresponding to λ, in particular, it is simple.*

Using the abstract approach of Subsection 3.5 we get the 2-representations of $\mathscr{S}$ and $\mathscr{S}^{\mathbb{Z}}$ corresponding to right cells. By construction, these 2-representations of $\mathscr{S}$ are (genuine) categorifications of the corresponding cell modules. Note that our terminology from Subsection 3.5 is opposite to the one above. This is done to compensate for the anti-nature of our categorification of $\mathbb{S}_n$ and $\mathbb{H}$ (see Corollary 5.6 and Theorem 8.2). Later on we will construct these 2-representations intrinsically in terms of the category $\mathcal{O}$.

8.4. **Categorification of the induced sign module.** Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra. Then the category $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ is stable with respect to the action of projective functors and hence defines, by restriction, a 2-representation of $\mathscr{S}$. The algebra $B^{\mathfrak{p}}_{\lambda}$ inherits from B_{λ} a positive grading. Hence we can consider the graded lift ${}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p}}_{\lambda}$ of $\mathcal{O}^{\mathfrak{p}}_{\lambda}$, which is stable with respect to the action of graded lifts of projective functors. In this way we obtain, by restriction, a 2-representation of $\mathscr{S}^{\mathbb{Z}}$. A natural question to ask is: What do these 2-representations categorify?

Example 8.8. If $\mathfrak{p} = \mathfrak{g}$, the category $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ is semi-simple and $L(\lambda)$ is the unique up to isomorphism simple module in this category. We have $\theta_s L(\lambda) = 0$ for every simple reflection s. Thus $[\mathcal{O}^{\mathfrak{p}}_{\lambda}]^{\mathbb{C}}$ is one-dimensional and we have $(e + s)[L(\lambda)] = 0$, that is $s[L(\lambda)] = -[L(\lambda)]$. This means that our 2-representation of $\mathscr{S}$ on $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ is a categorification of the *sign* W-module.

The above example suggests the answer in the general case. Let $\mathbb{H}^{\mathfrak{p}}$ denote the Hecke algebra of $W^{\mathfrak{p}}$. For $u \in \{-v, v^{-1}\}$ define on $\mathbb{Z}[v, v^{-1}]$ the structure of an $\mathbb{H}^{\mathfrak{p}}$-bimodule, which we denote by $\mathcal{V}_u$, via the surjection $H_s \mapsto u$ for every simple reflection $s \in W^{\mathfrak{p}}$. Define the *parabolic* $\mathbb{H}$-module $\mathcal{M}_u$ as follows:

$$\mathcal{M}_u := \mathcal{V}_u \otimes_{\mathbb{H}^{\mathfrak{p}}} \mathbb{H}.$$

Let $W(\mathfrak{p})$ denote the set of shortest coset representatives in $W^{\mathfrak{p}} \setminus W$. The elements $M_x = 1 \otimes H_x$, $x \in W(\mathfrak{p})$, form a basis of $\mathcal{M}_u$. The action of $\underline{H}_s$, $s \in S$, in this basis is given by:

$$M_x\underline{H}_s = \begin{cases} M_{xs} + vM_x, & xs \in W(\mathfrak{p}), xs > x; \\ M_{xs} + v^{-1}M_x, & xs \in W(\mathfrak{p}), xs < x; \\ (v + v^{-1})M_x, & xs \notin W(\mathfrak{p}), u = v^{-1}; \\ 0, & xs \notin W(\mathfrak{p}), u = -v. \end{cases} \tag{8.1}$$

Under the specialization $v \mapsto 1$, the module $\mathcal{M}_{v^{-1}}$ specializes to the permutation module ${}_{\mathbb{Z}[W^{\mathfrak{p}}]} \setminus \mathbb{Z}[W]$, while the module $\mathcal{M}_{-v}$ specializes to the *induced sign* module

$$\mathrm{sign} \bigotimes_{\mathbb{Z}[W^{\mathfrak{p}}]} \mathbb{Z}[W],$$

where the $\mathbb{Z}[W^{\mathfrak{p}}]$-module sign is given via the surjection $\mathbb{Z}[W^{\mathfrak{p}}] \to \mathbb{Z}$ which sends a simple reflection s to -1.

Applying $\mathrm{Z}_{\mathfrak{p}}$ to Lemma 7.9, we obtain:

Lemma 8.9. *For every $x \in W(\mathfrak{p})$ and any simple reflection s we have:*
(*a*) *If $xs \notin W(\mathfrak{p})$, then $\theta_s\Delta^{\mathfrak{p}}(x \cdot \lambda) = 0$.*
(*b*) *If $xs \in W(\mathfrak{p})$ and $xs > x$, then there is a short exact sequence as follows:*

$$0 \to \Delta^{\mathfrak{p}}(x \cdot \lambda)\langle -1\rangle \to \theta_s\Delta^{\mathfrak{p}}(x \cdot \lambda) \to \Delta^{\mathfrak{p}}(xs \cdot \lambda) \to 0.$$

(*c*) *If $xs \in W(\mathfrak{p})$ and $xs < x$, then there is a short exact sequence as follows:*

$$0 \to \Delta^{\mathfrak{p}}(xs \cdot \lambda) \to \theta_s\Delta^{\mathfrak{p}}(x \cdot \lambda) \to \Delta^{\mathfrak{p}}(x \cdot \lambda)\langle 1\rangle \to 0.$$

There is a unique $\mathbb{Z}[v, v^{-1}]$-linear isomorphism from $\mathcal{M}_{-v}$ to $[{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}]$ sending M_x to $[\Delta^{\mathfrak{p}}(x \cdot \lambda)]$ for all $x \in W(\mathfrak{p})$. Comparing (8.1) with Lemma 8.9, we obtain:

Proposition 8.10 ([So4]). *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}$ (resp. $\mathcal{O}_\lambda^{\mathfrak{p}}$) categorifies the parabolic module $\mathcal{M}_{-v}$ (resp. the induced sign module).*

Since $\mathcal{O}_\lambda^{\mathfrak{p}}$, being quasi-hereditary, has finite projective dimension, the $\mathbb{H}$-module $\mathcal{M}_{-v} \cong [{}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}]$ has the following natural bases:

- The *standard* basis M_x given by classes of parabolic Verma modules.
- The *Kazhdan-Lusztig* basis given by classes of indecomposable projective modules.
- The *dual Kazhdan-Lusztig* basis given by classes of simple modules.
- The *twisted Kazhdan-Lusztig* basis given by classes of indecomposable tilting modules.

8.5. **Categorification of Specht modules.** Similarly to the previous subsection, let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra and $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular. Denote by $w_o^{\mathfrak{p}}$ the longest element of $W_{\mathfrak{p}}$. Then the element $w_o^{\mathfrak{p}}w_o$ is the unique longest element in $W(\mathfrak{p})$. In [IS] it was shown (by explicit calculation) that there exists an integral $\lambda' \in \mathfrak{h}^*_{\mathrm{dom}}$ such that the category $\mathcal{O}_{\lambda'}^{\mathfrak{p}}$ has a unique simple module and hence is semi-simple. In particular, this unique simple module is both projective and injective. Using translation functors it follows that there is at least one projective injective module in $\mathcal{O}_\lambda^{\mathfrak{p}}$. In fact, we have the following:

Theorem 8.11 ([Ir1]). *For $w \in W(\mathfrak{p})$ the following conditions are equivalent:*
(*a*) *The module $P^{\mathfrak{p}}(w \cdot \lambda)$ is injective.*
(*b*) *The module $L^{\mathfrak{p}}(w \cdot \lambda)$ occurs in the socle of some $\Delta^{\mathfrak{p}}(\mu)$.*
(*c*) *The module $L^{\mathfrak{p}}(w \cdot \lambda)$ has maximal Gelfand-Kirillov dimension in $\mathcal{O}_{\lambda}^{\mathfrak{p}}$.*
(*d*) $w \sim_R w_o^{\mathfrak{p}} w_o$.

As a direct corollary we have that indecomposable projective injective modules in $\mathcal{O}_{\lambda}^{\mathfrak{p}}$ are indexed by elements of the right cell of $w_o^{\mathfrak{p}} w_o$. Since projective functors preserve both projective and injective modules, the latter suggests that their action on the category of projective-injective modules in $\mathcal{O}_{\lambda}^{\mathfrak{p}}$ should categorify the cell module corresponding to the right cell of $w_o^{\mathfrak{p}} w_o$.

Denote by $\mathcal{C}$ the Serre subcategory of $\mathcal{O}_{\lambda}^{\mathfrak{p}}$ consisting of all modules which do not have maximal Gelfand-Kirillov dimension. Let $\mathcal{R}_{\mathfrak{p}}$ denote the right cell of $w_o^{\mathfrak{p}} w_o$. By Theorem 8.11, $\mathcal{C}$ is generated by $L^{\mathfrak{p}}(w \cdot \lambda)$ such that $w \notin \mathcal{R}_{\mathfrak{p}}$. As projective functors are essentially tensoring with finite-dimensional modules, they preserve $\mathcal{C}$. Therefore we can restrict the action of $\mathscr{S}$ to $\mathcal{C}$ and also get the induced action of $\mathscr{S}$ on the quotient $\mathcal{O}_{\lambda}^{\mathfrak{p}}/\mathcal{C}$. If we denote by $e_{\mathfrak{p}}$ the sum of all primitive idempotents in $B_{\lambda}^{\mathfrak{p}}$, corresponding to $L(w \cdot \lambda)$, $w \in \mathcal{R}_{\mathfrak{p}}$, we get $\mathcal{O}_{\lambda}^{\mathfrak{p}}/\mathcal{C} \cong e_{\mathfrak{p}} B_{\lambda}^{\mathfrak{p}} e_{\mathfrak{p}}$-mod. Let $\mathcal{C}^{\mathbb{Z}}$ denote the graded lift of $\mathcal{C}$. Denote by ξ the transpose of the partition of n corresponding to $\mathfrak{p}$.

Theorem 8.12 ([KMS]). (*a*) *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on the additive category of projective-injective modules in ${}^{\mathbb{Z}}\mathcal{O}_{\lambda}^{\mathfrak{p}}$ (resp. $\mathcal{O}_{\lambda}^{\mathfrak{p}}$) categorifies the cell module $\mathbb{Z}[v, v^{-1}]\mathcal{R}_{\mathfrak{p}}$ (resp. the Specht module corresponding to ξ).*
(*b*) *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on ${}^{\mathbb{Z}}\mathcal{O}_{\lambda}^{\mathfrak{p}}/\mathcal{C}^{\mathbb{Z}}$ (resp. $\mathcal{O}_{\lambda}^{\mathfrak{p}}/\mathcal{C}$) categorifies the cell module $\mathbb{Z}[v, v^{-1}]\mathcal{R}_{\mathfrak{p}}$ (resp. the Specht module corresponding to ξ) after extending scalars to $\mathbb{Q}$.*

Idea of the proof. Compare the action of θ_s in the basis of indecomposable projective modules modules with the action of $\underline{H}_s$. □

We have the usual *Kazhdan-Lusztig* basis in the split Grothendieck group of the category of projective-injective modules in ${}^{\mathbb{Z}}\mathcal{O}_{\lambda}^{\mathfrak{p}}$ given by classes of indecomposable projective modules. The only obvious natural basis of $[\mathcal{O}_{\lambda}^{\mathfrak{p}}/\mathcal{C}]$ is the one given by classes of simple modules (the *dual Kazhdan-Lusztig* basis). The algebra $e_{\mathfrak{p}} B_{\lambda}^{\mathfrak{p}} e_{\mathfrak{p}}$ is known to be symmetric (see [MS3]), in particular, it has infinite global dimension in general. As a consequences, the classes of projective modules do not usually form a $\mathbb{Z}$-basis of $[\mathcal{O}_{\lambda}^{\mathfrak{p}}/\mathcal{C}]$.

Example 8.13. To categorify the trivial $\mathbb{S}_n$-module we have to take $\mathfrak{p} = \mathfrak{b}$. In this case $W_{\mathfrak{p}} = \{e\}$, $\mathcal{R}_{\mathfrak{p}} = \{w_o\}$ and $e_{\mathfrak{p}} B_{\lambda}^{\mathfrak{p}} e_{\mathfrak{p}} \cong C_{\lambda}$ is the coinvariant algebra. For $n > 1$ the class $[P(w_o \cdot \lambda)]$ does not form a $\mathbb{Z}$-basis in the Grothendieck group, but gives a basis if one extends scalars to $\mathbb{Q}$ or $\mathbb{C}$.

A partition ξ of n does not define the parabolic subalgebra $\mathfrak{p}$ uniquely, so one could naturally ask whether our categorification of the Specht module corresponding to ξ depends on the choice of $\mathfrak{p}$. The answer is given by the following:

Theorem 8.14 ([MS3]). *If $\mathfrak{p}$ and $\mathfrak{q}$ are two parabolic subalgebras which give rise to the same partition ξ, then the corresponding categorifications of the Specht modules are equivalent (in the sense of representations of 2-categories).*

The equivalence of Theorem 8.14 is provided by a certain composition of derived Zuckerman functors (which naturally commute with projective functors).

Remark 8.15. Let $W = \{e, s, t, st, ts, sts, tst, stst\}$ be of type B_2 and $V = \mathbb{C}$ be the one dimensional module on which s and t acts via 1 and -1, respectively. Then

V cannot be categorified via the action of the B_2-analogue of $\mathscr{S}$ on some module category. Indeed, for any such action θ_t must act by the zero functor and θ_s must act by a nonzero functor, which is not possible as s and t belong to the same two-sided cell. On the other hand, V admits a categorification using a triangulated category.

9. $\mathbb{S}_n$-categorification: (induced) cell modules

9.1. **Categories $\mathcal{O}^{\hat{\mathcal{R}}}$.** Let $\mathcal{R}$ be a right cell of W and $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular. Set

$$\hat{\mathcal{R}} = \{w \in W : w \leq_R x \text{ for any } x \in \mathcal{R}\}$$

and denote by $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ the Serre subcategory of $\mathcal{O}_\lambda$ generated by $L(w \cdot \lambda)$, $w \in \hat{\mathcal{R}}$.

Example 9.1. If $\mathcal{R}$ contains $w_o^{\mathfrak{p}} w_o$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, then $\mathcal{O}_\lambda^{\hat{\mathcal{R}}} = \mathcal{O}_\lambda^{\mathfrak{p}}$.

In the general case the associative algebra describing $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ might have infinite global dimension (in particular, it does not have to be quasi-hereditary). Denote by $\mathrm{Z}_{\mathcal{R}} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ the right exact functor of taking the maximal quotient contained in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$. This is a natural generalization of Zuckerman's functor. Similarly to the case of the parabolic category we will denote by $P^{\mathcal{R}}(w \cdot \lambda)$ etc. all structural modules in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$.

From the Kazhdan-Lusztig combinatorics it follows that $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ is closed with respect to the action of projective functors. Hence the restriction of the action of projective functors to $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ gives a 2-representation of $\mathscr{S}$. Let ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ denote the graded lift of $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$, which, obviously, exists. Then ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ carries the natural structure of a 2-representation of $\mathscr{S}^{\mathbb{Z}}$.

Proposition 9.2 ([MS4]). *For $w \in \hat{\mathcal{R}}$ the following conditions are equivalent:*
(*a*) *The module $P^{\mathcal{R}}(w \cdot \lambda)$ is injective.*
(*b*) *The module $L^{\mathcal{R}}(w \cdot \lambda)$ has maximal Gelfand-Kirillov dimension in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$.*
(*c*) *$w \in \mathcal{R}$.*

9.2. **Categorification of cell modules.** Denote by $\mathcal{C}$ the Serre subcategory of $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ consisting of all modules which do not have maximal Gelfand-Kirillov dimension. By Proposition 9.2, $\mathcal{C}$ is generated by $L^{\mathcal{R}}(w \cdot \lambda)$ such that $w \notin \mathcal{R}$ and is stable under the action of $\mathscr{S}$. Therefore we have the induced action of $\mathscr{S}$ on the quotient $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}/\mathcal{C}$. Let $\mathcal{C}^{\mathbb{Z}}$ denote the graded lift of $\mathcal{C}$.

Theorem 9.3 ([MS4]). (*a*) *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on the additive category of projective-injective modules in ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ (resp. $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$) categorifies the cell module $\mathbb{Z}[v, v^{-1}]\mathcal{R}$ (resp. its specialization for $v = 1$).*
(*b*) *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\hat{\mathcal{R}}}/\mathcal{C}^{\mathbb{Z}}$ (resp. $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}/\mathcal{C}^{\mathbb{Z}}$) categorifies the cell module $\mathbb{Z}[v, v^{-1}]\mathcal{R}$ (resp. its specialization for $v = 1$) after extending scalars to $\mathbb{Q}$.*

We have the usual *Kazhdan-Lusztig* basis in the split Grothendieck group of the category of projective-injective modules in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ given by classes of indecomposable projective modules. We also have the natural basis of $[\mathcal{O}_\lambda^{\hat{\mathcal{R}}}/\mathcal{C}]$ given by classes of simple modules (the *dual Kazhdan-Lusztig* basis).

Theorem 9.4 ([MS4]). *If $\mathcal{R}$ and $\mathcal{R}'$ are two right cells of W inside the same two-sided cell, then the corresponding 2-representations of $\mathscr{S}^{\mathbb{Z}}$ and $\mathscr{S}$ categorifying cell modules are equivalent.*

Idea of the proof. The equivalence is provided by composing functors Q_s, $s \in S$. □

Corollary 9.5 ([MS4]). *The endomorphism algebra of a basic projective injective module in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ does not depend on the choice of $\mathcal{R}$ inside a fixed two-sided cell. In particular, this endomorphism algebra is symmetric.*

Remark 9.6. The categorification of the cell module constructed in Theorem 9.3 is isomorphic to the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ of $\mathscr{S}$ in the sense of Subsection 3.5.

9.3. **Categorification of permutation modules.** Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular and $\nu \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral (but, possibly, singular). Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ such that $W_{\mathfrak{p}} = W_\nu$. Denote by $W[\mathfrak{p}]$ the set of *longest* representatives in cosets from $W_{\mathfrak{p}} \setminus W$. Let $\mathcal{C}$ denote the Serre subcategory of $\mathcal{O}_\lambda$ generated by $L(w\cdot\lambda)$, $w \in W \setminus W[\mathfrak{p}]$. We identify the quotient $\mathcal{O}_\lambda/\mathcal{C}$ with the full subcategory of $\mathcal{O}_\lambda$ consisting of all M having a two step presentation $P_1 \to P_0 \twoheadrightarrow M$, where both P_1 and P_0 are isomorphic to direct sums of projectives of the form $P(w\cdot\lambda)$, $w \in W[\mathfrak{p}]$. Then $\mathcal{O}_\lambda/\mathcal{C}$ can be described in terms of Harish-Chandra bimodules as follows:

Theorem 9.7 ([BG]). *Tensoring with $\Delta(\nu)$ induces an equivalence between ${}^\infty_\lambda\mathcal{H}^1_\nu$ and $\mathcal{O}_\lambda/\mathcal{C}$.*

The equivalence from Theorem 9.7 obviously commutes with the left action of projective functors. Therefore we may regard ${}^\infty_\lambda\mathcal{H}^1_\nu$ as a 2-representation of $\mathscr{S}$. Note that the category $\mathcal{C}$ clearly admits a graded lift $\mathcal{C}^{\mathbb{Z}}$, which allows us to consider a graded version $({}^\infty_\lambda\mathcal{H}^1_\nu)^{\mathbb{Z}}$ of ${}^\infty_\lambda\mathcal{H}^1_\nu$. The category $({}^\infty_\lambda\mathcal{H}^1_\nu)^{\mathbb{Z}}$ is a 2-representation of $\mathscr{S}^{\mathbb{Z}}$. If ν is singular, the category ${}^\infty_\lambda\mathcal{H}^1_\nu$ has infinite global dimension and hence is not described by a quasi-hereditary algebra.

For $w \in W[\mathfrak{p}]$ denote by $\Delta_{\mathfrak{p}}(w)$ the preimage in ${}^\infty_\lambda\mathcal{H}^1_\nu$ of the quotient of $P(w\cdot\lambda)$ by the trace of all $P(x\cdot\lambda)$ such that $x \in W[\mathfrak{p}]$ and $x < w$. We also let $P(w)$ denote the preimage of $P(w\cdot\lambda)$ and define $L(w)$ as the simple top of $P(w)$. Finally, denote by $\overline{\Delta}_{\mathfrak{p}}(w)$ the quotient of $\Delta_{\mathfrak{p}}(w)$ modulo the trace of $\Delta_{\mathfrak{p}}(w)$ in the radical of $\Delta_{\mathfrak{p}}(w)$.

Theorem 9.8 ([FKM2, KM]). *Let $w \in W[\mathfrak{p}]$.*

(*a*) *The kernel of the natural projection $P(w) \twoheadrightarrow \Delta_{\mathfrak{p}}(w)$ has a filtration whose subquotients are isomorphic to $\Delta_{\mathfrak{p}}(x)$ for $x < w$.*
(*b*) *The kernel of the natural projection $\overline{\Delta}_{\mathfrak{p}}(w) \twoheadrightarrow L(w)$ has a filtration whose subquotients are isomorphic to $L(x)$ for $x > w$.*
(*c*) *The module $\Delta_{\mathfrak{p}}(w)$ has a filtration whose subquotients are isomorphic to $\overline{\Delta}_{\mathfrak{p}}(w)$.*

Theorem 9.8 says that the associative algebra describing the category ${}^\infty_\lambda\mathcal{H}^1_\nu$ is *properly stratified* in the sense of [Dl]. Modules $\Delta_{\mathfrak{p}}(w)$ are called *standard modules* and modules $\overline{\Delta}_{\mathfrak{p}}(w)$ are called *proper standard modules* with respect to this structure. Note that standard modules have finite projective dimension, while proper standard modules usually have infinite projective dimension. There is an appropriate titling theory developed for properly stratified algebras in [AHLU]. In [FKM2] it is shown that ${}^\infty_\lambda\mathcal{H}^1_\nu$ is Ringel self-dual.

Lemma 9.9. *For every $x \in W[\mathfrak{p}]$ and any simple reflection s we have:*

(*a*) *If $xs \notin W(\mathfrak{p})$, then there is a short exact sequence as follows:*

$$0 \to \Delta_{\mathfrak{p}}(x)\langle -1\rangle \to \theta_s\Delta_{\mathfrak{p}}(x) \to \Delta_{\mathfrak{p}}(x)\langle 1\rangle \to 0.$$

(*b*) *If $xs \in W[\mathfrak{p}]$ and $xs > x$, then there is a short exact sequence as follows:*

$$0 \to \Delta_{\mathfrak{p}}(x)\langle -1\rangle \to \theta_s\Delta_{\mathfrak{p}}(x) \to \Delta_{\mathfrak{p}}(xs) \to 0.$$

(*c*) *If $xs \in W[\mathfrak{p}]$ and $xs < x$, then there is a short exact sequence as follows:*

$$0 \to \Delta_{\mathfrak{p}}(xs) \to \theta_s \Delta_{\mathfrak{p}}(x) \to \Delta_{\mathfrak{p}}(x)\langle 1\rangle \to 0.$$

There is a unique $\mathbb{Z}[v, v^{-1}]$-linear homomorphism from $\mathcal{M}_{v^{-1}}$ to $[({}^{\infty}_{\lambda}\mathcal{H}^1_{\nu})^{\mathbb{Z}}]$ sending M_x to $[\Delta_{\mathfrak{p}}(x)]$ for all $x \in W(\mathfrak{p})$. This becomes an isomorphism if we extend our scalars to $\mathbb{Q}$. Comparing (8.1) with Lemma 9.9, we obtain:

Proposition 9.10 ([MS1])**.** *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on $({}^{\infty}_{\lambda}\mathcal{H}^1_{\nu})^{\mathbb{Z}}$ (resp. ${}^{\infty}_{\lambda}\mathcal{H}^1_{\nu}$) categorifies the parabolic module $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{M}_{v^{-1}}$ (resp. the corresponding permutation module). The integral version $\mathcal{M}_{v^{-1}}$ is categorified by the action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on the additive category of projective modules in $({}^{\infty}_{\lambda}\mathcal{H}^1_{\nu})^{\mathbb{Z}}$ (resp. ${}^{\infty}_{\lambda}\mathcal{H}^1_{\nu}$).*

The Grothendieck group of $({}^{\infty}_{\lambda}\mathcal{H}^1_{\nu})^{\mathbb{Z}}$ has two natural bases:

- The *proper standard* basis given by classes of proper standard modules.
- The *dual Kazhdan-Lusztig* basis given by classes of simple modules.

Extending the scalar to $\mathbb{Q}$ we get three other bases:

- The *standard* basis given by classes of standard modules.
- The *Kazhdan-Lusztig* basis given by classes of indecomposable projective modules.
- The *twisted Kazhdan-Lusztig* basis given by classes of indecomposable tilting modules.

9.4. **Parabolic analogues of $\mathcal{O}$.** Comparing our categorifications of the induced sign module (Subsection 8.4) and the permutation module (Subsection 9.3) one could observe certain similarities in constructions. A natural question is: could they be two special cases of some more general construction. The answer turns out to be "yes" and the corresponding general construction is the general approach to parabolic generalizations of $\mathcal{O}$ proposed in [FKM1].

Let λ and $\mathfrak{p}$ be as in the previous section. Denote by $\mathfrak{n}$ and $\mathfrak{a}$ the nilpotent radical and the Levi quotient of $\mathfrak{p}$, respectively. Then $\mathfrak{a}$ is a reductive Lie algebra, isomorphic to the direct sum of some $\mathfrak{gl}_{k_i}$. For any category $\mathcal{C}$ of $\mathfrak{a}$-modules one can consider the full subcategory $\mathcal{O}(\mathfrak{p}, \mathcal{C})$ of the category of all $\mathfrak{g}$-modules, which consists of all modules M satisfying the following conditions:

- M is finitely generated;
- the action of $U(\mathfrak{n})$ on M is locally finite;
- after restriction to $\mathfrak{a}$, the module M decomposes into a direct sum of modules from $\mathcal{C}$.

Under some rather general assumptions on $\mathcal{C}$ one can show that the category $\mathcal{O}(\mathfrak{p}, \mathcal{C})$ admits a block decomposition such that each block is equivalent to the module category over a finite dimensional associative algebra (see [FKM1]). Here we will deal with some very special categories $\mathcal{C}$ and will formulate all results for these categories.

Fix some right cell $\mathcal{R}$ in $W_{\mathfrak{p}}$ and consider the Serre subcategory $\mathcal{C}_{\mathcal{R}}$ of the category $\mathcal{O}$ for the algebra $\mathfrak{a}$, which is generated by simple modules $L(w \cdot \lambda_{\mathfrak{a}})$ and all their translations to all possible walls, where $\lambda_{\mathfrak{a}}$ is integral, regular and dominant and $w \in \hat{\mathcal{R}}$. Denote by $\mathcal{C}'_{\mathcal{R}}$ the Serre subcategory of $\mathcal{C}_{\mathcal{R}}$ generated by simple modules $L(w \cdot \lambda_{\mathfrak{a}})$ and all their translations to all possible walls, where $\lambda_{\mathfrak{a}}$ is integral, regular and dominant and $w \in \hat{\mathcal{R}} \setminus \mathcal{R}$. By Subsection 9.2, the action of projective functors on a regular block of $\mathcal{C}_{\mathcal{R}}/\mathcal{C}'_{\mathcal{R}}$ categorifies the cell $W_{\mathfrak{p}}$-module corresponding to $\mathcal{R}$. As usual, we identify the quotient $\mathcal{C} := \mathcal{C}_{\mathcal{R}}/\mathcal{C}'_{\mathcal{R}}$ with the full subcategory of $\mathcal{C}_{\mathcal{R}}$ consisting of modules which have a presentation by projective-injective modules (see Proposition 9.2).

Now we can consider the category $\mathcal{O}(\mathfrak{p},\mathcal{C})$, as was proposed in [MS4]. To obtain our categorification of the induced sign module we have to take $\mathcal{R}=\{e\}$, which implies that $\mathcal{C}_{\mathcal{R}}$ is the category of all finite dimensional semi-simple integral $\mathfrak{a}$-modules and $\mathcal{C}'_{\mathcal{R}}=0$. To obtain our categorification of the permutation module we have to take $\mathcal{R}=\{w_o^{\mathfrak{p}}\}$, which implies that each regular block of $\mathcal{C}_{\mathcal{R}}/\mathcal{C}'_{\mathcal{R}}$ is the category of modules over the coinvariant algebra of $W_{\mathfrak{p}}$, realized via modules in $\mathcal{O}$ (for $\mathfrak{a}$) admitting a presentation by projective-injective modules. By construction, $\mathcal{O}(\mathfrak{p},\mathcal{C})$ is a Serre subquotient of $\mathcal{O}$, in particular, it inherits from $\mathcal{O}$ a decomposition into blocks, indexed by dominant λ.

Lemma 9.11 ([MS4])**.** *Let $\lambda\in\mathfrak{h}^*_{\mathrm{dom}}$ be integral and regular. Denote by $\mathcal{K}$ and $\mathcal{K}'$ the Serre subcategories of $\mathcal{O}_\lambda$ generated by $L(xy\cdot\lambda)$, where $y\in W(\mathfrak{p})$ and $x\in\hat{\mathcal{R}}$ or $x\in\hat{\mathcal{R}}\setminus\mathcal{R}$, respectively. Then $\mathcal{O}(\mathfrak{p},\mathcal{C})_\lambda$ is equivalent to $\mathcal{K}/\mathcal{K}'$.*

For $w\in\mathcal{R}W(\mathfrak{p})\subset W$ denote by $P^{\mathfrak{p},\mathcal{R}}(w)$ the projective cover of $L(w\cdot\lambda)$ in $\mathcal{K}/\mathcal{K}'$. For $x,x'\in\mathcal{R}$ and $y,y'\in W(\mathfrak{p})$ we write $xy\preceq x'y'$ if and only if $y\leq y'$. Denote by $\Delta^{\mathfrak{p},\mathcal{R}}(w)$ the quotient of $P^{\mathfrak{p},\mathcal{R}}(w)$ modulo the trace of all projectives $P^{\mathfrak{p},\mathcal{R}}(w')$ such that $w'\prec w$. Denote by $\overline{\Delta}^{\mathfrak{p},\mathcal{R}}(w)$ the quotient of $\Delta^{\mathfrak{p},\mathcal{R}}(w)$ modulo the trace of $\Delta^{\mathfrak{p},\mathcal{R}}(w')$ such that $w'\preceq w$ in the radical of $\Delta^{\mathfrak{p},\mathcal{R}}(w)$. Denote by $L^{\mathfrak{p},\mathcal{R}}(w)$ the unique simple top of $P^{\mathfrak{p},\mathcal{R}}(w)$. The $\mathfrak{g}$-module $L(w\cdot\lambda)$ is a representative for the simple object $L^{\mathfrak{p},\mathcal{R}}(w)$ of $\mathcal{K}/\mathcal{K}'$. We can now formulate the following principal structural properties of the category $\mathcal{K}/\mathcal{K}'$.

Theorem 9.12 ([MS4])**.** *Let $w\in\mathcal{R}W(\mathfrak{p})$.*

(*a*) *The kernel of the natural projection $P^{\mathfrak{p},\mathcal{R}}(w)\twoheadrightarrow\Delta^{\mathfrak{p},\mathcal{R}}(w)$ has a filtration whose subquotients are isomorphic to $\Delta^{\mathfrak{p},\mathcal{R}}(x)$ for $x\prec w$.*

(*b*) *The kernel of the natural projection $\overline{\Delta}^{\mathfrak{p},\mathcal{R}}(w)\twoheadrightarrow L^{\mathfrak{p},\mathcal{R}}(w)$ has a filtration whose subquotients are isomorphic to $L^{\mathfrak{p},\mathcal{R}}(x)$ for $w\prec x$.*

(*c*) *The module $\Delta^{\mathfrak{p},\mathcal{R}}(w)$ has a filtration whose subquotients are isomorphic to $\overline{\Delta}^{\mathfrak{p},\mathcal{R}}(x)$ such that $w\preceq x$ and $x\preceq w$.*

Theorem 9.12 says that the algebra describing $\mathcal{O}(\mathfrak{p},\mathcal{C})_\lambda$ is *standardly stratified* in the sense of [CPS2]. Modules $\Delta^{\mathfrak{p},\mathcal{R}}(w)$ and $\overline{\Delta}^{\mathfrak{p},\mathcal{R}}(w)$ are called *standard* and *proper standard* modules with respect to this structure, respectively. There is an appropriate titling theory developed for standardly stratified algebras in [Fr]. In [MS4] it is shown that $\mathcal{O}(\mathfrak{p},\mathcal{C})_\lambda$ is Ringel self-dual and that for this category one has an analogue of Theorem 8.11.

Clearly, the category $\mathcal{O}(\mathfrak{p},\mathcal{C})_\lambda$ admits a graded analogue, $\mathcal{O}(\mathfrak{p},\mathcal{C})^{\mathbb{Z}}_\lambda$.

9.5. **Categorification of induced cell modules.** Let $\mathcal{R}$ be as in the previous subsection. By definition, the induced cell module $\mathbb{Z}[v,v^{-1}]\mathcal{R}\otimes_{\mathbb{H}^{\mathfrak{p}}}\mathbb{H}$ has a basis, which consists of all elements of the form $\underline{H}_x\otimes H_y$, where $x\in\mathcal{R}$ and $y\in W(\mathfrak{p})$ (see [HY] for detailed combinatorics of induced cell modules). Define a $\mathbb{Z}[v,v^{-1}]$-linear map Ψ from $\mathbb{Z}[v,v^{-1}]\mathcal{R}\otimes_{\mathbb{H}^{\mathfrak{p}}}\mathbb{H}$ to $[\mathcal{O}(\mathfrak{p},\mathcal{C})^{\mathbb{Z}}_\lambda]$ by sending $\underline{H}_x\otimes H_y$ to the class of $\Delta^{\mathfrak{p},\mathcal{R}}(xy)$. As both $\mathcal{K}$ and $\mathcal{K}'$, as defined in the previous subsection, are closed with respect to the action of projective functors, we have a natural action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on the additive category of projective objects in $\mathcal{O}(\mathfrak{p},\mathcal{C})^{\mathbb{Z}}_\lambda$ (resp. $\mathcal{O}(\mathfrak{p},\mathcal{C})_\lambda$).

Theorem 9.13 ([MS4])**.** (*a*) *After extending the scalars to $\mathbb{Q}$, the map Ψ becomes an isomorphism of $\mathbb{H}$-modules.*

(*b*) *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on the additive category of projective objects in $\mathcal{O}(\mathfrak{p},\mathcal{C})^{\mathbb{Z}}_\lambda$ (resp. $\mathcal{O}(\mathfrak{p},\mathcal{C})_\lambda$) is a categorification of $\mathbb{Z}[v,v^{-1}]\mathcal{R}\otimes_{\mathbb{H}^{\mathfrak{p}}}\mathbb{H}$ (resp. the corresponding specialized induced cell module).*

(c) *The action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on $\mathcal{O}(\mathfrak{p},\mathcal{C})^{\mathbb{Z}}_{\lambda}$ (resp. $\mathcal{O}(\mathfrak{p},\mathcal{C})_{\lambda}$) is a categorification of $\mathbb{Z}[v,v^{-1}]\mathcal{R}\otimes_{\mathbb{H}^{\mathfrak{p}}}\mathbb{H}$ (resp. the corresponding specialized induced cell module) after extending scalars to $\mathbb{Q}$.*

Idea of the proof. One should check that the action of projective functors on standard modules is compatible with the action of $\underline{H}_s$ on the module $\mathbb{Z}[v,v^{-1}]\mathcal{R}\otimes_{\mathbb{H}^{\mathfrak{p}}}\mathbb{H}$ in the standard basis $\underline{H}_x\otimes H_y$. This reduces either to short exact sequences similar to the ones in Lemma 8.9(b)-(c), or to the action of the $\mathfrak{a}$-analogue of θ_s on projective modules in $\mathcal{C}$. That the latter action is the right one follows from Theorem 9.3. □

The $\mathbb{H}$-module $[\mathcal{O}(\mathfrak{p},\mathcal{C})^{\mathbb{Z}}_{\lambda}]$ has two natural bases:

- The *proper standard* basis given by classes of proper standard modules.
- The *dual Kazhdan-Lusztig* basis given by classes of simple modules.

After extending the scalars to $\mathbb{Q}$, we get three other natural bases:

- The *standard* basis given by classes of standard modules.
- The *Kazhdan-Lusztig* basis given by classes of indecomposable projective modules.
- The *twisted Kazhdan-Lusztig* basis given by classes of indecomposable tilting modules.

Similarly to the case of cell modules, we have the following uniqueness result:

Proposition 9.14 ([MS4])**.** *The categorification of the module $\mathbb{Z}[v,v^{-1}]\mathcal{R}\otimes_{\mathbb{H}^{\mathfrak{p}}}\mathbb{H}$ described in Theorem 9.13 does not depend, up to equivalence, on the choice of $\mathcal{R}$ inside a fixed two-sided cell.*

10. Category $\mathcal{O}$: Koszul duality

10.1. **Quadratic dual of a positively graded algebra.** Let $A=\oplus_{i\in\mathbb{Z}}A_i$ be a *positively graded* $\mathbb{C}$-algebra, that is:

- for all $i<0$ we have $A_i=0$;
- for all $i\in\mathbb{Z}$ we have $\dim A_i<\infty$;
- for some $k\in\mathbb{N}$ we have $A_0\cong\mathbb{C}^k$.

Denote by A-gfmod the category of locally finite dimensional graded A-modules. Let $\overline{A}$ denote the subalgebra of A, generated by A_0 and A_1. Clearly, $\overline{A}$ inherits from A a positive grading. The multiplication in A defines on A_1 the structure of an A_0-A_0–bimodule. Consider the *free tensor algebra* $A_0[A_1]$ of this bimodule. It is defined as follows:

$$A_0[A_1]:=\bigoplus_{i\geq 0}A_1^{\otimes i},$$

where $A_1^{\otimes 0}:=A_0$, for $i>0$ we have $A_1^{\otimes i}=A_1\otimes_{A_0}A_1\otimes_{A_0}\cdots\otimes_{A_0}A_1$ (with i factors A_1), and multiplication is given, as usual, by tensoring over A_0. The identity maps on A_0 and A_1 induce an algebra homomorphism $A_0[A_1]\twoheadrightarrow A$, whose image coincides with $\overline{A}$. Let I be the kernel of this homomorphism. The algebra $A_0[A_1]$ is graded in the natural way ($A_1^{\otimes i}$ has degree i) and I is a homogeneous ideal of $A_0[A_1]$. Hence $I=\oplus_{i\in\mathbb{Z}}I_i$.

Definition 10.1. The algebra A is called *quadratic* if $A=\overline{A}$ and I is generated in degree 2.

Example 10.2. Any path algebra of a quiver without relations is graded in the natural way (each arrow has degree one) and is obviously quadratic. Another example of a quadratic algebra is the algebra D of dual numbers.

Consider the dual A_0-A_0–bimodule $A_1^*=\mathrm{Hom}_{\mathbb{C}}(A_1,\mathbb{C})$. Since $\mathbb{C}$ is symmetric, we may identify $(A_1^*)^{\otimes i}$ with $(A_1^{\otimes i})^*$.

Definition 10.3. The *quadratic dual* of A is defined as the algebra

$$A^! := A_0[A_1^*]/(I_2^*),$$

where $I_2^* := \{f : A_0[A_1]_2 \to \mathbb{C} \text{ such that } f|_{I_2} = 0\}$ (under the above identification).

Directly from the definition we have that $(A^!)^! \cong A$ in case A is quadratic.

Example 10.4. The quadratic dual of $\mathbb{C}[x]/(x^2)$ is $\mathbb{C}[x]$ and the quadratic dual of $\mathbb{C}[x]$ is $\mathbb{C}[x]/(x^2)$. On the other hand, for any $k > 2$ the quadratic dual of $\mathbb{C}[x]/(x^k)$ is $\mathbb{C}[x]/(x^2)$ (note that in this case $\mathbb{C}[x]/(x^k)$ is not quadratic). The quadratic dual of the path algebra of a quiver without relations is the path algebra of the opposite quiver with radical square zero.

10.2. **Linear complexes of projectives.** Denote by $\mathcal{LC}(A)$ the category of *linear complexes* of projective A-modules defined as follows: objects of $\mathcal{LC}(A)$ are complexes

$$\mathcal{X}^\bullet : \qquad \dots \to X^{i-1} \to X^i \to X^{i+1} \to \dots$$

such that for any $i \in \mathbb{Z}$ we have $X^i \in \mathrm{add}(A\langle i\rangle)$; morphisms in $\mathcal{LC}(A)$ are just usual morphisms between complexes of graded modules.

Theorem 10.5 ([MVS, MO2, MOS])**.** *The category $\mathcal{LC}(A)$ is equivalent to the category $A^!$-gfmod of locally finite dimensional graded $A^!$-modules.*

Let $e_{\mathtt{i}}$, $\mathtt{i} \in \Lambda$, be a complete and irredundant list of primitive idempotents for A. Then $A_0 = \oplus_{\mathtt{i}\in\Lambda}\mathbb{C}\langle e_{\mathtt{i}}\rangle$. For $\mathtt{i} \in \Lambda$ denote by $\mathcal{P}^\bullet(\mathtt{i})$ a minimal projective resolution of the simple graded A-module $e_{\mathtt{i}}A_0$. Since A is positively graded, every indecomposable direct summand of $\mathcal{P}^i(\mathtt{i})$, $i \in \mathbb{Z}$, is isomorphic to $Ae_{\mathtt{j}}\langle j\rangle$ for some $\mathtt{j} \in \Lambda$ and $j \leq i$. Taking all summands of the form $Ae_{\mathtt{j}}\langle i\rangle$ produces a subcomplex $\overline{\mathcal{P}}^\bullet(\mathtt{i})$ of $\mathcal{P}^\bullet(\mathtt{i})$. Similarly one defines injective (co)resolutions $\mathcal{I}^\bullet(\mathtt{i})$ and their linear parts $\overline{\mathcal{I}}^\bullet(\mathtt{i})$. Denote by $\circledast$ the graded duality. Then we have the Nakayama functor $\mathrm{N} := \mathrm{Hom}_A({}_-, A)^\circledast$, which induces an equivalence between the additive categories of graded projective and injective A-modules.

Proposition 10.6 ([MO2, MOS])**.** (*a*) *Every simple object of $\mathcal{LC}(A)$ is isomorphic, up to a shift of the form $\langle s\rangle[-s]$, $s \in \mathbb{Z}$, to $Ae_{\mathtt{i}}$, considered as a complex concentrated in position zero.*

(*b*) *Every indecomposable injective object of $\mathcal{LC}(A)$ is isomorphic, up to a similar shift, to $\overline{\mathcal{P}}^\bullet(\mathtt{i})$ for some $\mathtt{i} \in \Lambda$.*

(*c*) *Every indecomposable projective object of $\mathcal{LC}(A)$ is isomorphic, up to a similar shift, to $\mathrm{N}^{-1}\,\overline{\mathcal{I}}^\bullet(\mathtt{i})$ for some $\mathtt{i} \in \Lambda$.*

Denote by $\mathfrak{Q}$ the full subcategory of $\mathcal{LC}(A)$ with objects being all $\mathrm{N}^{-1}\,\overline{\mathcal{I}}^\bullet(\mathtt{i})$, $\mathtt{i} \in \Lambda$, and their shifts as above. Then, by Theorem 10.5, $\mathfrak{Q}$ can be regarded as a complex of A-$A^!$-bimodules and from Proposition 10.6 it follows that this complex is projective on both sides. For any category $\mathcal{A}$, whose objects are complexes of graded modules, denote by $\mathcal{A}^\downarrow$ and $\mathcal{A}^\uparrow$ the full subcategories of $\mathcal{A}$ whose nonzero components are concentrated inside the corresponding regions as shown on Figure 1 (see [MOS] for explicit formulae). With this notation we have the following result proved in [MOS]:

Theorem 10.7 (Quadratic duality)**.** (*a*) *There is a pair of adjoint functors as follows:*

$$\mathcal{D}^\downarrow(A\text{-gfmod}) \underset{\mathrm{K}' := \mathfrak{Q}\overset{\mathcal{L}}{\otimes}_{A^!}{}_-}{\overset{\mathrm{K} := \mathcal{R}\mathrm{Hom}_A(\mathfrak{Q},{}_-)}{\rightleftarrows}} \mathcal{D}^\uparrow(A^!\text{-gfmod})$$

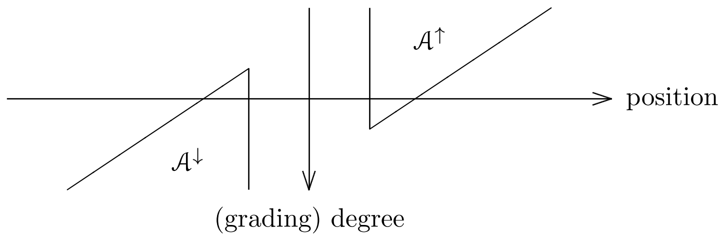


FIGURE 1. The supports of objects from the categories $\mathcal{A}^{\downarrow}$ and $\mathcal{A}^{\uparrow}$.

(*b*) K *maps simple* A*-modules to injective* $A^!$*-modules and* K$'$ *maps simple* $A^!$*-modules to projective* A*-modules.*
(*c*) *For all* $i,j \in \mathbb{Z}$ *we have* $\mathrm{K} \circ \langle j\rangle \circ [i] = \langle -j\rangle \circ [i+j] \circ \mathrm{K}$.

The functors K and K$'$ are called *quadratic duality* functors.

10.3. **Koszul duality.** In a special case Theorem 10.7 can be improved.

Definition 10.8. A positively graded algebra A is called Koszul provided that $\overline{\mathcal{P}}^{\bullet}(\mathtt{i}) = \mathcal{P}^{\bullet}(\mathtt{i})$ for every $\mathtt{i} \in \Lambda$.

Typical examples of Koszul algebras are $\mathbb{C}[x]$, $\mathbb{C}[x]/(x^2)$ and quiver algebras without relations.

Theorem 10.9 (Koszul duality)**.** (*a*) *The functors* K *and* K$'$ *from Theorem 10.7 are mutually inverse equivalences of categories if and only if* A *is Koszul.*
(*b*) *If* A *is a finite dimensional Koszul algebra of finite global dimension, then the functors* K *and* K$'$ *restrict to an equivalence between* $\mathcal{D}^b(A\text{-gmod})$ *and* $\mathcal{D}^b(A^!\text{-gmod})$.

In the case of a Koszul algebra A the functors K and K$'$ are called *Koszul duality* functors.

Corollary 10.10. *If* A *is Koszul, then* $(A^!)^{\mathrm{op}}$ *is isomorphic to the Yoneda algebra* $\mathrm{Ext}_A^*(A_0, A_0)$.

The algebra $(A^!)^{\mathrm{op}}$ is called the *Koszul dual* of A. Koszul algebras and duals were introduced in [Pr]. The Koszul duality theorem as an equivalence of derived categories was established in [BGS]. For the category $\mathcal{O}$ we have the following:

Theorem 10.11 ([So1, BGS])**.** *Let* $\mathfrak{p}$ *be a parabolic subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{b}$, $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ *be integral and regular, and* $\mu \in \mathfrak{h}^*_{\mathrm{dom}}$ *be integral such that* $W_\mu = W_{\mathfrak{p}}$. *Then both* $B^{\mathfrak{p}}_\lambda$ *and* B_μ *are Koszul, moreover, they are Koszul dual to each other. In particular,* B_λ *is Koszul and Koszul self-dual.*

The Koszul self-duality of B_λ does not preserve primitive idempotents but rather acts on their indexing set W as the map $w \mapsto w^{-1}w_o$. Theorem 10.11 is generalized to arbitrary categories ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ in [Bac1]. An approach to the Koszul dualities on $\mathcal{O}$ via linear complexes was proposed in [Ma4] and was further explored in [MOS, Ma2, Ma3]. There is a well-developed theory of Koszul duality for abstract quasi-hereditary algebras, see [ADL, MO2, Ma5, Ma6].

Instead of the category $\mathcal{LC}(A)$ one could alternatively consider the category of linear complexes of injective A-modules. In the case of quasi-hereditary algebra A one can also consider the category of linear complexes of tilting A-modules. In the case of $\mathcal{O}_\lambda$ all these categories are equivalent via T_{w_o} (or C_{w_o}).

Many structural modules in $\mathcal{O}_\lambda$ admit a linear projective or a linear injective or a linear tilting "resolutions". For example (see [Ma2]):

- Simple modules admit both linear injective, projective and tilting resolutions.
- Standard (Verma) modules admit linear projective and tilting resolutions.
- Costandard modules admit linear injective and tilting resolutions.
- Projective modules in $\mathcal{O}_\lambda^{\mathfrak{p}}$ admit linear projective and tilting resolutions (in $\mathcal{O}_\lambda$).
- Tilting modules in $\mathcal{O}_\lambda^{\mathfrak{p}}$ admit linear tilting resolutions (in $\mathcal{O}_\lambda$).

As one could expect after Proposition 10.6, all these resolutions play some role in the structure of the corresponding category of linear complexes. For example, linear tilting resolutions of simple modules are tilting objects in the category of linear complexes of tilting modules. In fact, the category of linear complexes of tilting modules seems to be the most "symmetric" one.

10.4. **Koszul dual functors.** Let A be a finite dimensional Koszul algebra of finite global dimension and $\mathrm{K} : \mathcal{D}^b(A\text{-gmod}) \to \mathcal{D}^b(A^!\text{-gmod})$ the corresponding Koszul duality functor. Let F be an endofunctor of $\mathcal{D}^b(A\text{-gmod})$ and G an endofunctor of $\mathcal{D}^b(A^!\text{-gmod})$.

Definition 10.12. The functors F and G are called *Koszul dual* provided that the following diagram commutes up to an isomorphism of functors:

$$\begin{array}{ccc} \mathcal{D}^b(A\text{-gmod}) & \xrightarrow{\quad\mathrm{K}\quad} & \mathcal{D}^b(A^!\text{-gmod}) \\ {\scriptstyle \mathrm{F}}\downarrow & & \downarrow{\scriptstyle \mathrm{G}} \\ \mathcal{D}^b(A\text{-gmod}) & \xrightarrow{\quad\mathrm{K}\quad} & \mathcal{D}^b(A^!\text{-gmod}) \end{array}$$

For the category $\mathcal{O}$ we have the following pairs of Koszul dual functors:

Theorem 10.13 ([RH, MOS])**.** *Under the identification of $B_\lambda^{\mathfrak{p}}$ and $B_\mu^!$, given by Theorem 10.11, the following functors (all corresponding to a fixed simple reflection s) are Koszul dual to each other (up to an appropriate shift in grading and position):*

(*a*) *Translation to the wall and derived Zuckerman functor.*
(*b*) *Translation out of the wall and canonical inclusion.*
(*c*) *Derived shuffling and derived twisting.*
(*d*) *Derived coshuffling and derived completion.*

The statements of Theorem 10.13(a)-(b) were conjectured in [BGS] and first proved in [RH] using dg-categories. In [MOS] one finds a unifying approach to all statements based on the study of the category $\mathcal{LC}(B_\lambda)$.

10.5. **Alternative categorification of the permutation module.** As we know from Proposition 8.10, the action of $\mathscr{S}^{\mathbb{Z}}$ (resp. $\mathscr{S}$) on ${}^{\mathbb{Z}}\mathcal{O}_\lambda^{\mathfrak{p}}$ (resp. $\mathcal{O}_\lambda^{\mathfrak{p}}$) categorifies the parabolic module $\mathcal{M}_{-v}$ (resp. the induced sign module). Since all our functors are exact, we can derive all involved categories and get the same result.

The point in deriving the picture is that we can do more with the derived picture. Namely, we may now apply the Koszul duality functor K to obtain an action of $\mathscr{S}^{\mathbb{Z}}$ on the bounded derived category of graded modules over the Koszul dual of $B_\lambda^{\mathfrak{p}}$. By Theorem 10.11, the Koszul dual of $B_\lambda^{\mathfrak{p}}$ is B_μ, where $\mu \in \mathfrak{h}^*_{\text{dom}}$ is integral and such that $W_\mu = W_{\mathfrak{p}}$.

Now we have to be careful, since K does not obviously commute with shifts (both in position and grading). Instead, we have the formula of Theorem 10.7(c). Because of our conventions, this formula implies that K induces an automorphism ψ of $\mathbb{H}$ given by $v \mapsto -v^{-1}$. This means that the image, under the Koszul duality functor K, of the $\mathbb{H}$-module $\mathcal{M}_{-v}$ becomes the $\mathbb{H}$-module $\mathcal{M}_{v^{-1}}$. Taking Theorem 10.13 into account, we obtain:

Proposition 10.14. *Let μ be as above. The action of derived Zuckerman functors on $\mathcal{D}^b(B_\mu\text{-gmod})$ categorifies (in the image of the Kazhdan-Lusztig basis under ψ) the parabolic module $\mathcal{M}_{v^{-1}}$.*

Taking into account the relation between twisting and Zuckerman functors from Proposition 6.7, the naïve picture of the corresponding categorification of the permutation module can be described as follows:

Corollary 10.15. *Let μ be as above. The action of derived twisting functors on $\mathcal{D}^b(B_\mu\text{-gmod})$ gives a naïve categorification of the permutation module W_μ in the standard basis of W.*

In many cases, studying some categorification picture, it might be useful to switch to the Koszul dual (if it exists), where computation might turn out to be much easier (for example, some derived functors might become exact after taking the Koszul dual).

11. $\mathfrak{sl}_2$-categorification: simple finite dimensional modules

11.1. **The algebra $U_v(\mathfrak{sl}_2)$.** Recall that $\mathbb{N}_0$ denotes the set of non-negative integers. Denote by $\mathbb{C}(v)$ the field of rational functions with complex coefficients in an indeterminate v. Recall the following definition (see e.g. [Ja3]):

Definition 11.1. The quantum group $U_v(\mathfrak{sl}_2)$ is the associative $\mathbb{C}(v)$-algebra with generators E, F, K, K^{-1} and relations

$$KE = v^2EK,\ KF = v^{-2}FK,\ KK^{-1} = K^{-1}K = 1,\ EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

The algebra $U_v(\mathfrak{sl}_2)$ is a Hopf algebra with comultiplication Δ given by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}.$$

For $a \in \mathbb{Z}$ set

$$[a] := \frac{v^a - v^{-a}}{v - v^{-1}}$$

and for all $n \in \mathbb{N}$ put

$$[n]! := [1][2]\cdots[n].$$

Set $[0]! = 1$ and define

$$\begin{bmatrix} a \\ n \end{bmatrix} := \frac{[a][a-1]\cdots[a-n+1]}{[n]!}.$$

Define also

$$E^{(a)} := \frac{E^a}{[a]!}, \qquad F^{(a)} := \frac{F^a}{[a]!}.$$

Proposition 11.2 ([Ja3])**.** (a) *The algebra $U_v(\mathfrak{sl}_2)$ has no zero divisors.*

(b) *The algebra $U_v(\mathfrak{sl}_2)$ has a PBW basis consisting of monomials $F^sK^nE^r$, where $r, s \in \mathbb{N}_0$ and $n \in \mathbb{Z}$.*

(c) *The center of $U_v(\mathfrak{sl}_2)$ is generated by the* quantum Casimir element

$$C := FE + \frac{Kv + K^{-1}v^{-1}}{(v - v^{-1})^2}.$$

11.2. **Finite dimensional representations of $U_v(\mathfrak{sl}_2)$.** For $n \in \mathbb{N}_0$ let $\mathcal{V}_n$ and $\hat{\mathcal{V}}_n$ denote the $\mathbb{C}(v)$-vector space with basis $\{w_0, w_1, \dots, w_n\}$. Define linear operators E, F, $K^{\pm 1}$ on $\mathcal{V}_n$ using the following formulae:

$$Ew_k = [k+1]w_{k+1}, \quad Fw_k = [n-k+1]w_{k-1}, \quad K^{\pm 1}w_k = v^{\pm(2k-n)}w_k \tag{11.1}$$

(under the convention $w_{-1} = w_{n+1} = 0$). Define linear operators E, F, $K^{\pm 1}$ on $\hat{\mathcal{V}}_n$ using the following formulae:

$$Ew_k = [k+1]w_{k+1}, \quad Fw_k = -[n-k+1]w_{k-1}, \quad K^{\pm 1}w_k = -v^{\pm(2k-n)}w_k. \tag{11.2}$$

By a direct calculation we get the following:

Lemma 11.3. *Formulae* (11.1) *and* (11.2) *define on both $\mathcal{V}_n$ and $\hat{\mathcal{V}}_n$ the structure of a simple $U_v(\mathfrak{sl}_2)$-module of dimension $n+1$.*

For $n = 1$ the module $\mathcal{V}_1$ is called the *natural* $U_v(\mathfrak{sl}_2)$-module.

Theorem 11.4 ([Ja3])**.** (*a*) *For every $n \in \mathbb{N}_0$ there is a unique simple $U_v(\mathfrak{sl}_2)$-module of dimension $n+1$, namely $\mathcal{V}_n$, on which K acts semi-simply with powers of v as eigenvalues.*

(*b*) *For every $n \in \mathbb{N}_0$ there is a unique simple $U_v(\mathfrak{sl}_2)$-module of dimension $n+1$, namely $\hat{\mathcal{V}}_n$, on which K acts semi-simply with minus powers of v as eigenvalues.*

(*c*) *Every simple $U_v(\mathfrak{sl}_2)$-module of dimension $n+1$ is isomorphic to either $\mathcal{V}_n$ or $\hat{\mathcal{V}}_n$ and these two modules are not isomorphic.*

(*d*) *Every finite-dimensional $U_v(\mathfrak{sl}_2)$-module on which K acts semi-simply is completely reducible.*

Let X, Y be two $U_v(\mathfrak{sl}_2)$-modules. Using the comultiplication Δ we can turn $X \otimes Y$ into a $U_v(\mathfrak{sl}_2)$-module. As usual, for $n \in \mathbb{N}_0$ we denote by $X^{\otimes n}$ the product $\underbrace{X \otimes X \otimes \dots \otimes X}_{n \text{ factors}}$.

Proposition 11.5. (*a*) *For $n \in \mathbb{N}_0$ we have:*

$$\mathcal{V}_1 \otimes \mathcal{V}_n \cong \begin{cases} \mathcal{V}_1, & n = 0; \\ \mathcal{V}_{n-1} \oplus \mathcal{V}_{n+1}, & \textit{otherwise}. \end{cases}$$

(*b*) *For every $n \in \mathbb{N}_0$ the module $\mathcal{V}_n$ appears in $\mathcal{V}_1^{\otimes n}$ with multiplicity one and all other irreducible direct summands of $\mathcal{V}_1^{\otimes n}$ are isomorphic to $\mathcal{V}_k$ for some $k < n$.*

11.3. **Categorification of $\mathcal{V}_1^{\otimes n}$.** Let $n \in \mathbb{N}_0$. Set $\mathfrak{gl}_0 := 0$ (then $U(\mathfrak{gl}_0) = \mathbb{C}$). Consider the algebra $\mathfrak{gl}_n$. For $i = 0, 1, 2, \dots, n$ denote by λ_i the dominant integral weight such that $\lambda_i + \rho = (1, 1, \dots, 1, 0, 0, \dots, 0)$, where the number of 1's equals i. The weight λ_i is singular and $W_{\lambda_i} \cong \mathbb{S}_i \oplus \mathbb{S}_{n-i}$ is the parabolic subgroup of $\mathbb{S}_n$ corresponding to permutations of the first i and the last $n-i$ elements. Consider the categories

$$\mathfrak{V}_n := \bigoplus_{i=0}^{n} \mathcal{O}_{\lambda_i} \quad \text{and} \quad \mathfrak{V}_n^{\mathbb{Z}} := \bigoplus_{i=0}^{n} \mathcal{O}_{\lambda_i}^{\mathbb{Z}}.$$

Directly from the definition we have:

Lemma 11.6. *For $i = 0, 1, \dots, n$ the Grothendieck group $[\mathcal{O}_{\lambda_i}^{\mathbb{Z}}]$ is a free $\mathbb{Z}[v, v^{-1}]$-module of rank $\binom{n}{i}$. In particular, $[\mathfrak{V}_n^{\mathbb{Z}}]$ is a free $\mathbb{Z}[v, v^{-1}]$-module of rank 2^n.*

Let $V := \mathbb{C}^n$ be the natural representation of $\mathfrak{gl}_n$. For $i = 0, 1, \dots, n-1$ denote by E_i the projective functor from $\mathcal{O}_{\lambda_i}$ to $\mathcal{O}_{\lambda_{i+1}}$ given by tensoring with V and then projecting onto $\mathcal{O}_{\lambda_{i+1}}$. Define the endofunctors E and F of $\mathfrak{V}_n$ as follows:

$$\mathrm{E} := \bigoplus_{i=0}^{n-1} \mathrm{E}_i, \qquad \mathrm{F} := \bigoplus_{i=0}^{n-1} \mathrm{E}_i^*$$

(our convention is that $\mathrm{E}\mathcal{O}_{\lambda_n} = 0$, $\mathrm{F}\mathcal{O}_{\lambda_0} = 0$ and $\mathrm{E}_n = 0$). Set $\mathrm{F}_{i+1} := \mathrm{E}_i^*$. By a direct calculation on the level of the Grothendieck group one easily shows the following:

Proposition 11.7. *The action of* E *and* F *on* $\mathfrak{V}_n$ *gives a weak categorification of the* n*-th tensor power of the natural* $\mathfrak{gl}_2$*-module.*

We would like, however, to upgrade this at least to the level of the quantum algebra. For this we have to carefully define graded lifts for our functors E_i. For $i = 0, 1, \dots, n-1$ consider the integral dominant weights $\mu_i = (2, 2, \dots, 2, 1, 0, 0, \dots, 0) - \rho$ and $\nu_i = (2, 2, \dots, 2, 0, 0, 0, \dots, 0) - \rho$, where in both cases the number of 2's equals i. Then the functor E_i can be realized as the composition of the following projective functors:

(11.3)

$$\mathcal{O}_{\lambda_i} \xrightarrow{\theta_{\lambda_i,\nu_i}} \mathcal{O}_{\nu_i} \xrightarrow{\theta_{\nu_i,s_{i+1}\cdot\mu_i}} \mathcal{O}_{\mu_i} \xrightarrow{\theta_{\mu_i,\nu_{i+1}}} \mathcal{O}_{\nu_{i+1}} \xrightarrow{\theta_{\nu_{i+1},\lambda_{i+1}}} \mathcal{O}_{\lambda_{i+1}}$$

Here the first and the last functors are equivalences, the second functor is translation out of the wall and the third functor is translation onto the wall (for $i = n-1$ the second functor disappears and E_i is just translation to the wall).

Consider now usual graded lifts of all involved categories. Let equivalences in (11.3) be lifted as equivalences of degree zero. Use Theorem 7.5 to lift translation to and out of the wall such that they correspond to restriction and induction for the associated algebras of invariants in the coinvariants. This determines a graded lift of E_i, which we will denote by the same symbol, abusing notation. This defines graded lifts for functors E and F from the above, which we again will denote by the same symbols. Finally, define K_i as the endofunctor $\langle n-2i\rangle$ of $\mathcal{O}^{\mathbb{Z}}_{\lambda_i}$ and set

$$\mathrm{K} := \bigoplus_{i=0}^{n} \mathrm{K}_i.$$

Theorem 11.8 (Categorification of $\mathcal{V}_1^{\otimes n}$, [BFK, St3, FKS])**.** (a) *The functors* E, F *and* K *satisfy the relations*

$$\mathrm{KE} \cong \mathrm{EK}\langle -2\rangle, \quad \mathrm{KF} \cong \mathrm{FK}\langle 2\rangle, \quad \mathrm{KK}^{-1} \cong \mathrm{K}^{-1}\mathrm{K} \cong \mathrm{Id}.$$

(b) *For* $i = 0, 1, \dots, n$ *there are isomorphisms*

$$\mathrm{E}_{i-1}\mathrm{F}_i \oplus \bigoplus_{r=1}^{n-i-1} \mathrm{Id}\langle 2r+2i+1-n\rangle \cong \mathrm{F}_{i+1}\mathrm{E}_i \oplus \bigoplus_{r=1}^{i-1} \mathrm{Id}\langle n+2r+1-2i\rangle.$$

(c) *In the Grothendieck group we have the equality*

$$(v - v^{-1})([\mathrm{E}_{i-1}][\mathrm{F}_i] - [\mathrm{F}_{i+1}][\mathrm{E}_i]) = [\mathrm{K}_i] - [\mathrm{K}_i^{-1}]$$

(d) *The functor* $\mathrm{E}_{i-1}\mathrm{F}_i$ *is a summand of* $\mathrm{F}_{i+1}\mathrm{E}_i$ *if* $n - 2i > 0$*, the functor* $\mathrm{F}_{i+1}\mathrm{E}_i$ *is a summand of* $\mathrm{E}_{i-1}\mathrm{F}_i$ *if* $n - 2i < 0$.

(e) *There is an isomorphism of* $U_v(\mathfrak{sl}_2)$*-modules,* $[\mathfrak{V}_n^{\mathbb{Z}}] \cong \mathcal{V}_1^{\otimes n}$*, where the action of* $U_v(\mathfrak{sl}_2)$ *on the left hand side is given by the exact functors* E, F *and* $\mathrm{K}^{\pm 1}$.

Idea of the proof. Use classification of projective functors and check all equalities on the level of the Grothendieck group. □

From Theorem 11.8 we see that the module $\mathcal{V}_1^{\otimes n}$ comes equipped with the following natural bases:

- the *standard basis* given by classes of Verma modules;
- the *twisted canonical basis* given by classes of indecomposable projective modules;
- the *dual canonical basis* given by classes of simple modules;

- the *canonical basis* given by classes of indecomposable tilting modules.

The canonical basis can be defined, similarly to Kazhdan-Lusztig basis, as the set of self-dual elements with respect to some anti-involution. The dual of the canonical basis is the dual in the sense of a certain bilinear form, which is categorified via the usual Ext-form twisted by T_{w_o}.

Example 11.9. In the case $n = 0$ we have $\mathcal{O}_{\lambda_0} = \mathbb{C}$-mod and both F and E are zero functors. The functor K is just the identity.

Example 11.10. In the case $n = 1$ we have $\mathcal{O}_{\lambda_0} = \mathcal{O}_{\lambda_1} = \mathbb{C}$-mod and both F and E are the identity functors between the two parts of $\mathfrak{V}_1$. The functor K acts as $\langle -1\rangle$ on the part annihilated by E and as $\langle 1\rangle$ on the other part.

Example 11.11. In the case $n = 2$ we have $\mathcal{O}_{\lambda_0} = \mathcal{O}_{\lambda_2} = \mathbb{C}$-mod and $\mathcal{O}_{\lambda_1}$ is the category of modules over the algebra from Subsection 4.7. The functors E_0 and E_1 are translations out of the wall and onto the wall, respectively.

11.4. **Categorification of $\mathcal{V}_n$.** The weight spaces of the categorified finite dimensional $U_v(\mathfrak{sl}_2)$-module $\mathcal{V}_1^{\otimes n}$ are certain singular blocks of the category $\mathcal{O}$. The action of $U_v(\mathfrak{sl}_2)$ on this categorified picture is given by projective functors. Every singular block of $\mathcal{O}$ contains a unique (up to isomorphism) indecomposable projective injective module. Projective functors preserve (the additive category of) projective injective modules. Therefore we may restrict our $U_v(\mathfrak{sl}_2)$-action to this category, which also induces an $U_v(\mathfrak{sl}_2)$-action on the corresponding abelianization. This suggests the following:

Denote by $\hat{\mathfrak{V}}_n^{\mathbb{Z}}$ the full subcategory of $\mathfrak{V}_n^{\mathbb{Z}}$ consisting of all modules which do not have maximal Gelfand-Kirillov dimension (alternatively, whose Gelfand-Kirillov dimension is strictly smaller than the Gelfand-Kirillov dimension of a Verma module). Then $\hat{\mathfrak{V}}_n^{\mathbb{Z}}$ is a Serre subcategory of $\mathfrak{V}_n^{\mathbb{Z}}$ generated by all simples having not maximal Gelfand-Kirillov dimension (i.e. simples which are not Verma modules).

Theorem 11.12 (Categorification of $\mathcal{V}_n$, [FKS])**.** (*a*) *The functorial action of $U_v(\mathfrak{sl}_2)$ on $\mathfrak{V}_n^{\mathbb{Z}}$ preserves $\hat{\mathfrak{V}}_n^{\mathbb{Z}}$ and induces a functorial $U_v(\mathfrak{sl}_2)$-action on $\mathfrak{V}_n^{\mathbb{Z}}/\hat{\mathfrak{V}}_n^{\mathbb{Z}}$.*
(*b*) *The functorial $U_v(\mathfrak{sl}_2)$-action on $\mathfrak{V}_n^{\mathbb{Z}}/\hat{\mathfrak{V}}_n^{\mathbb{Z}}$ categorifies the $U_v(\mathfrak{sl}_2)$-module $\mathcal{V}_n$.*

Idea of the proof. Being direct summands of tensoring with finite dimensional modules, projective functors do not increase Gelfand-Kirillov dimension. This implies claim (a). Claim (b) follows from claim (a) and Theorem 11.8 by comparing characters. □

Note that the weight spaces of $\mathcal{V}_n$ are categorified as module categories over certain algebras of invariants in coinvariants. The latter have geometric interpretation as cohomology algebras of certain flag varieties. Hence all the above categorification pictures may be interpreted geometrically, see [FKS] for details.

The categorification of $\mathcal{V}_n$ constructed in Theorem 11.12 equips $\mathcal{V}_n$ with two natural bases:

- the *canonical basis* given by classes of indecomposable projective modules;
- the *dual canonical basis* given by classes of simple modules.

Example 11.13. In the case $n = 2$ we have $\hat{\mathfrak{V}}_2^{\mathbb{Z}} \subset \mathcal{O}_{\lambda_1}$. From Example 11.11 we already know that $\mathcal{O}_{\lambda_0} = \mathcal{O}_{\lambda_2} = \mathbb{C}$-mod. The category $\mathcal{O}_{\lambda_1}/\hat{\mathfrak{V}}_n^{\mathbb{Z}}$ is equivalent to the module category of the algebra D of dual numbers. The functors E_0 and E_1 are translations out of the wall and onto the wall, respectively (see Example 11.11).

In [FKS] one can find how to categorify tensor products of arbitrary (finite) collections of simple finite-dimensional $U_v(\mathfrak{sl}_2)$-modules.

11.5. **Koszul dual picture.** Using the Koszul duality functor from Subsection 10.3 we obtain categorifications of both $\hat{\mathcal{V}}_1^{\otimes n}$ and $\hat{\mathcal{V}}_n$ in terms of the parabolic category $\mathcal{O}$. The fact that the original action of the $U_v(\mathfrak{sl}_2)$-functor K, given by the shift of grading, transforms via the Koszul duality to a simultaneous shift in grading and position is the reason why the resulting modules is isomorphic to $\hat{\mathcal{V}}_1^{\otimes n}$ and not $\mathcal{V}_1^{\otimes n}$. Here we briefly describe how this goes.

To categorify $\hat{\mathcal{V}}_1^{\otimes n}$ consider the direct sum $\mathfrak{U}_n$ of all maximal parabolic subcategories in the regular block $\mathcal{O}_0$ for $\mathfrak{gl}_n$, that is

$$\mathfrak{U}_n := \bigoplus_{i=0}^{n} \mathcal{O}_0^{\mathfrak{p}_i}, \qquad \mathfrak{U}_n^{\mathbb{Z}} := \bigoplus_{i=0}^{n} {}^{\mathbb{Z}}\mathcal{O}_0^{\mathfrak{p}_i},$$

where $\mathfrak{p}_i$ is the parabolic subalgebra of $\mathfrak{g}$ such that the corresponding parabolic subgroup $W_{\mathfrak{p}_i}$ is isomorphic to $\mathbb{S}_i \oplus \mathbb{S}_{n-1}$, where the first component permutes the first i elements and the second components permutes the last $n-i$ elements. By Theorem 10.11, the category $\mathfrak{U}_n^{\mathbb{Z}}$ is the Koszul dual of the category $\mathfrak{V}_n^{\mathbb{Z}}$.

Using Theorem 10.13, the functorial action of $U_v(\mathfrak{sl}_2)$ on $\mathfrak{V}_n^{\mathbb{Z}}$ by projective functors transfers, via Koszul duality, to a functorial action of $U_v(\mathfrak{sl}_2)$ on $\mathcal{D}^b(\mathfrak{U}_n^{\mathbb{Z}})$ by derived Zuckerman functors. This gives us the Koszul dual categorification of $\hat{\mathcal{V}}_1^{\otimes n}$.

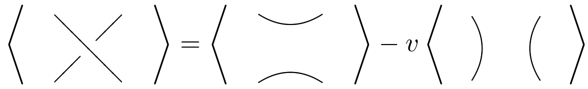
The Koszul dual categorification of the simple module $\hat{\mathcal{V}}_n$ is identified inside $\mathcal{D}^b(\mathfrak{U}_n^{\mathbb{Z}})$ as the full triangulated subcategory generated by simple finite dimensional modules (with all their shifts).

12. Application: categorification of the Jones polynomial

12.1. **Kauffman bracket and Jones polynomial.** Let L be a diagram of an oriented link. Let n_+ and n_- denote the number of *right* and *left* crossing in L, respectively, as shown on the following picture:

right crossing left crossing

The *Kauffman bracket* $\langle L \rangle \in \mathbb{Z}[v, v^{-1}]$ of L is defined via the following rule:

(12.1) 

together with $\langle \bigcirc L \rangle = (v + v^{-1})\langle L \rangle$ and normalized by the conditions $\langle \emptyset \rangle = 1$.

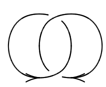
The (unnormalized) *Jones polynomial* $\hat{\mathrm{J}}(L)$ of L is defined by

$$\hat{\mathrm{J}}(L) := (-1)^{n_-} v^{n_+ - 2n_-} \langle L \rangle \in \mathbb{Z}[v, v^{-1}]$$

and the usual Jones polynomial $\mathrm{J}(L)$ is defined via $(v + v^{-1})\mathrm{J}(L) = \hat{\mathrm{J}}(L)$ (we use the normalization from [BN]).

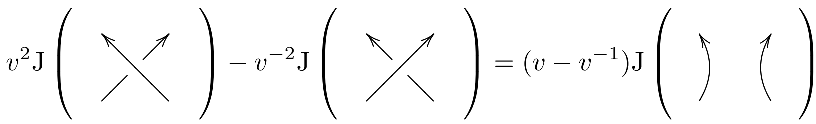
Theorem 12.1 ([Jn])**.** *The polynomial* $\mathrm{J}(L)$ *is an invariant of an oriented link.*

Example 12.2. For the *Hopf link*

$H :=$ 

we have $\hat{\mathrm{J}} = (v + v^{-1})(v + v^5)$ and $\mathrm{J}(H) = v + v^5$.

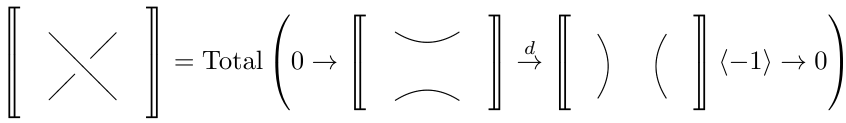
Proposition 12.3. *The Jones polynomial is uniquely determined by the property* $\mathrm{J}(\bigcirc) = 1$ *and the* skein relation

(12.2) 

12.2. **Khovanov's idea for categorification of** $\mathrm{J}(L)$**.** Khovanov's original idea ([Kv2]) how to categorify $\mathrm{J}(L)$ (as explained in [BN]) was to "upgrade" Kauffman's bracket to a new bracket $[\![\cdot]\!]$, which takes values in complexes of graded complex vector space (this works over $\mathbb{Z}$ as well, Khovanov originally works over $\mathbb{Z}[c]$ where $\deg c = 2$). Denote by V the two-dimensional graded vector space such that the nonzero homogeneous parts are of degree 1 and -1. The normalization conditions are easy to generalize:

$$[\![\varnothing]\!] = 0 \to \mathbb{C} \to 0, \qquad [\![\bigcirc L]\!] = V \otimes [\![L]\!] \,.$$

To "categorify" the rule (12.1) Khovanov proposes the axiom



for a certain differential d, the definition of which is the key ingredient of the construction. We refer the reader to [Kv2, BN] for details on how this works.

Appearance of complexes and taking total complexes in the above construction suggests possibility of its functorial interpretation (by action of functors on certain categories of complexes). We are going to describe this functorial approach below.

12.3. **Quantum $\mathfrak{sl}_2$-link invariants.** A usual way to construct knot and link invariants is using their connection to the braid group $\mathbb{B}_n$ given by the following theorem:

Theorem 12.4 (Alexander's Theorem)**.** *Every link can be obtained as the closure of a braid.*

The correspondence between links and braids given by Alexander's Theorem is not bijective, that is, different braids could give, upon closure, isotopic links. However, there is an easy way to find out when two different braids give rise to same links. For $n \in \mathbb{N}$ let $\sigma_1, \dots, \sigma_{n-1}$ be the generators of $\mathbb{B}_n$, and $\mathfrak{i}_n : \mathbb{B}_n \hookrightarrow \mathbb{B}_{n+1}$ the natural inclusion given by the identity on the σ_i's for $i = 1, \dots, n-1$. Denote by $\mathbb{B}$ the disjoint union of all $\mathbb{B}_n$, $n \geq 1$.

Theorem 12.5 (Markov's Theorem)**.** *Denote by* $\sim$ *the smallest equivalence relation on* $\mathbb{B}$ *which contains the conjugation relations on all* $\mathbb{B}_n$*,* $n \geq 1$*, and such that for any* $n \in \mathbb{N}$ *and any* $w \in \mathbb{B}_n$ *we have* $w \sim \mathfrak{i}_n(w)\sigma_n^{\pm 1}$*. Then two braids* $w, w' \in \mathbb{B}$ *give, by closing, isotopic links if and only if* $w \sim w'$*.*

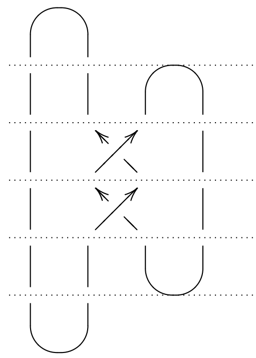
The closure of a braid can be obtained as a composition of the following "elementary" diagrams:

(12.3) 

the cup diagram the cap diagram right crossing left crossing

Example 12.6. The Hopf link from Example 12.2 can be obtained by the following sequence of elementary diagrams:

(12.4)



The idea of a quantum $\mathfrak{sl}_2$-invariant (originated in [RT]) is to associate to any elementary diagram a homomorphism of tensor powers of the two-dimensional $U_v(\mathfrak{sl}_2)$-module $\hat{\mathcal{V}}_1$. The module $\hat{\mathcal{V}}_1$ has basis $\{w_0, w_1\}$ as described in Subsection 11.2. It is convenient to denote the basis vector $w_{i_1} \otimes w_{i_2} \otimes \cdots \otimes w_{i_k}$, $i_1, i_2, \dots, i_k \in \{0,1\}$, of some tensor power of $\hat{\mathcal{V}}_1$ simply by $\mathtt{i}_1\mathtt{i}_2\cdots\mathtt{i}_k$, the so-called 0-1 *sequence.*

- To the cup diagram we associate the homomorphism $\cup : \mathbb{C}(v) \to \hat{\mathcal{V}}_1^{\otimes 2}$ defined as follows:
$$1 \mapsto \mathtt{01} + v\mathtt{10}.$$
- To the cap diagram we associate the homomorphism $\cap : \hat{\mathcal{V}}_1^{\otimes 2} \to \mathbb{C}(v)$ defined as follows:
$$\mathtt{00} \mapsto 0, \quad \mathtt{11} \mapsto 0, \quad \mathtt{01} \mapsto v^{-1}, \quad \mathtt{10} \mapsto 1.$$
- To the right crossing we associate the homomorphism $\hat{\mathcal{V}}_1^{\otimes 2} \to \hat{\mathcal{V}}_1^{\otimes 2}$ defined as follows:
$$\mathtt{00} \mapsto -v\mathtt{00}, \quad \mathtt{11} \mapsto -v\mathtt{11}, \quad \mathtt{01} \mapsto \mathtt{10} + (v^{-1} - v)\mathtt{01}, \quad \mathtt{10} \mapsto \mathtt{01}.$$
- To the left crossing we associate the homomorphism $\hat{\mathcal{V}}_1^{\otimes 2} \to \hat{\mathcal{V}}_1^{\otimes 2}$ defined as follows:
$$\mathtt{00} \mapsto -v^{-1}\mathtt{00}, \quad \mathtt{11} \mapsto -v^{-1}\mathtt{11}, \quad \mathtt{01} \mapsto \mathtt{10}, \quad \mathtt{10} \mapsto \mathtt{01} + (v - v^{-1})\mathtt{10}.$$

Using horizontal composition in the tensor category of finite dimensional $U_v(\mathfrak{sl}_2)$-modules the above induces morphisms $\cup_{i,n} : \hat{\mathcal{V}}_1^{\otimes n} \to \hat{\mathcal{V}}_1^{\otimes n+2}$ (inserting $\cup$ between the i-th and the $i+1$-st factors) ans similarly for the cap and crossing diagrams.

Given now an oriented link L, written as a composition of the elementary diagrams as above, we can read it (from bottom to top) as an endomorphism φ_L of $\hat{\mathcal{V}}_0$. Applied to 1, this endomorphism produces a Laurent polynomial in v.

Theorem 12.7 ([RT])**.** *The polynomials* $(-1)^{n_-}v^{n_+-2n_-}\varphi_L(1)$ *and* $\hat{\mathrm{J}}(L)$ *coincide.*

Idea of the proof. First check the skein relation (12.2) for $(-1)^{n_-}v^{n_+-2n_-}\varphi_L(1)$. Then check that our assignments for the cup and cap diagrams define the correct normalization. $\square$

Example 12.8. For the diagram L of the Hopf link as in (12.4), here is the step-by-step calculation of $\varphi_L(1)$ (the diagram is read from bottom to top):

$$\begin{aligned} 1 &\mapsto \mathtt{01} + v\mathtt{10} \\ &\mapsto \mathtt{0101} + v\mathtt{0110} + v\mathtt{1001} + v^2\mathtt{1010} \\ &\mapsto \mathtt{0011} - v^2\mathtt{0110} - v^2\mathtt{1001} + v^2\mathtt{1100} + v^2(v^{-1} - v)\mathtt{1100} \\ &\mapsto \mathtt{0101} + (v^{-1} - v)\mathtt{0011} + v^3\mathtt{0110} + v^3\mathtt{1001} + v^2(v^{-1} - v)\mathtt{1100} \\ &\quad\ + v^2((v^{-1} - v)^2 + 1)\mathtt{1010} \\ &\mapsto (v^{-1} + v^3)\mathtt{01} + v^2((v^{-1} - v)^2 + 2)\mathtt{10} \\ &\mapsto v^{-2} + 1 + v^2 + v^4. \end{aligned}$$

We thus have $\varphi_L(1) = v^{-2}(v + v^{-1})(v + v^5)$, which can be compared with $\hat{\mathrm{J}}(L)$, given by Example 12.2.

12.4. **Functorial quantum $\mathfrak{sl}_2$-link invariants.** From Subsection 11.5 we know a categorical realization of $\hat{\mathcal{V}}_1^{\otimes n}$ via the action of derived Zuckerman functors on the (bounded derived category of) the direct sum of maximal parabolic subcategories in the principal block $\mathcal{O}_0$ for $\mathfrak{gl}_n$. To categorify quantum $\mathfrak{sl}_2$-link invariants we thus only need to find a functorial interpretation for the cup, cap and crossing morphisms defined in the previous section. There is a catch though, since the cup and cap morphisms work between different tensor powers of $\hat{\mathcal{V}}_1$, so we have also to relate maximal parabolic subcategories in the principal block of $\mathcal{O}_0$ for $\mathfrak{gl}_k$ with $k \leq n$ to the case of $\mathfrak{gl}_n$.

To construct functorial homomorphisms between our categorical tensor powers of $\hat{\mathcal{V}}_1$ we have a good candidate: projective functors. Here we see the full advantage of the Koszul dual picture. In this picture the action of $U_v(\mathfrak{sl}_2)$ is given by derived Zuckerman functors, which commute with projective functors (see Proposition 6.1). Hence projective functors would automatically induces homomorphisms between our categorical tensor powers of $\hat{\mathcal{V}}_1$. This observation also suggests a solution to the catch. Among other projective functors, we have translations onto and out of the walls. Therefore one could try to relate maximal parabolic subcategories in certain singular blocks of $\mathcal{O}$ for $\mathfrak{gl}_n$ to maximal parabolic subcategories in regular blocks of $\mathcal{O}$ for $\mathfrak{gl}_k$ with $k < n$. This relation turns out to be known under the name *Enright-Shelton equivalence*, [ES].

For $i = 0, 1, \dots, n$ let $\mathfrak{p}_i$ be as in Subsection 11.5. For an integral $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ set

$$\mathcal{O}^{\max}_\lambda := \bigoplus_{i=0}^{n} \mathcal{O}^{\mathfrak{p}_i}_\lambda, \qquad {}^{\mathbb{Z}}\mathcal{O}^{\max}_\lambda := \bigoplus_{i=0}^{n} {}^{\mathbb{Z}}\mathcal{O}^{\mathfrak{p}_i}_\lambda.$$

Note that $\mathcal{O}^{\max}_\lambda$ might be zero (for example, if $n > 2$ and λ is most singular).

Proposition 12.9 ([ES])**.** *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be integral.*

(*a*) *The category $\mathcal{O}^{\max}_\lambda \neq 0$ if and only if every direct summand of W_λ is isomorphic to either $\mathbb{S}_1$ or $\mathbb{S}_2$.*

(*b*) *If the category $\mathcal{O}^{\max}_\lambda \neq 0$ and W_λ contains exactly k direct summands isomorphic to $\mathbb{S}_2$, then the category $\mathcal{O}^{\max}_\lambda$ is equivalent to the category $\mathcal{O}^{\max}_0$ for the algebra $\mathfrak{gl}_{n-2k}$.*

To proceed we observe that the closure of a braid necessarily has an even number of strands. This reduces our picture to the case of even n. For every parabolic subgroup $W_\mathfrak{p} \subset W$ satisfying the condition of Proposition 12.9(a) we fix some integral dominant $\lambda_{W_\mathfrak{p}}$ such that $W_\mathfrak{p} = W_{\lambda_{W_\mathfrak{p}}}$. Consider the categories

$$\mathfrak{W}_n := \bigoplus_{W_\mathfrak{p} \subset W} \mathcal{O}^{\max}_{\lambda_{W_\mathfrak{p}}}, \qquad \mathfrak{W}^{\mathbb{Z}}_n := \bigoplus_{W_\mathfrak{p} \subset W} {}^{\mathbb{Z}}\mathcal{O}^{\max}_{\lambda_{W_\mathfrak{p}}}.$$

By Proposition 12.9(b) and Subsection 11.5 the algebra $U_v(\mathfrak{sl}_2)$ acts on $\mathcal{D}^b(\mathfrak{W}_n^{\mathbb{Z}})$ via derived Zuckerman functors categorifying the corresponding direct sum of tensor powers of $\mathcal{V}_1$. To the elementary diagrams from (12.3) we associate the following endofunctors of $\mathcal{D}^b(\mathfrak{W}_n)$ and $\mathcal{D}^b(\mathfrak{W}_n^{\mathbb{Z}})$:

- To the cup diagram we associate translation out of the respective wall.
- To the cap diagram we associate translation to the respective wall.
- To the right crossing we associate the corresponding derived shuffling functor.
- To the left crossing we associate the corresponding derived coshuffling functor.

Note that all these functors commute with Zuckerman functors and hence define functorial homomorphisms between the corresponding categorified $U_v(\mathfrak{sl}_2)$-modules. In the graded picture we can adjust the gradings of our functors so that they correspond to the combinatorics of the morphisms associated to elementary diagrams as described in the previous subsection.

For $n = 2k$ let $W' \subset W$ be the unique parabolic subgroup isomorphic to the direct sum of k copies of $\mathbb{S}_2$. Then $\mathcal{O}^{\max}_{\lambda_{W'}}$ is semisimple an contains a unique simple module, which we denote by N. Reading our closed diagram L of an oriented link from bottom to top defines an endofunctor $\mathscr{F}_L$ of ${}^{\mathbb{Z}}\mathcal{O}^{\max}_{\lambda_{W'}}$. The following theorem was conjectured in [BFK] and proved in [St3]:

Theorem 12.10 (Categorification of the Jones polynomial)**.** *Let L be a diagram of an oriented link.*

(a) *The endofunctor $\mathscr{F}_L\langle 3n_+\rangle[-n_+]$ is an invariant of an oriented link.*
(b) *We have $\hat{\mathrm{J}}(L) = [\mathscr{F}_L\langle 2n_- - n_+\rangle[n_-]\,N]$.*

Idea of the proof. Derived shuffling and coshuffling functors are cones of adjunction morphisms between the identity functor and the corresponding translations through walls. This may be interpreted in terms of the skein relation (12.2). The normalization of $\mathscr{F}_L$ is easily computed using the combinatorics of translations through walls. There are many technical details which are checked by a careful analysis of the general combinatorics of translation functors. □

Remark 12.11. (a) Theorem 12.10 generalizes in a straightforward way to give invariants of oriented tangles (this, in particular, removes the restriction on n to be even), see [St3].
(b) Theorem 12.10 gives a functorial invariant of oriented tangles (see also [Kv2] for a different approach), which is a stronger invariant than the Jones polynomial (the latter is just roughly the Euler characteristics of the complex obtained by applying the functorial invariant).
(c) It is known (see [St4], based on [Br]) that Khovanov's categorification of the Jones polynomial is equivalent to the invariants constructed in Theorem 12.10.
(d) Theorem 12.10 can be extended to functorial quantum $\mathfrak{sl}_k$-invariants, see [Su, MS6], and to colored Jones polynomial, see [FSS].

13. $\mathfrak{sl}_2$-categorification: of Chuang and Rouquier

13.1. **Genuine $\mathfrak{sl}_2$-categorification.** In this section we describe some results from [CR]. Consider the Lie algebra $\mathfrak{sl}_2$ with the standard basis $\mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\Bbbk$ be a field.

Definition 13.1 ([CR])**.** An $\mathfrak{sl}_2$-categorification (over $\Bbbk$) is a tuple $(\mathcal{A}, \mathrm{E}, \mathrm{F}, q, a, \chi, \tau)$ such that:

(i) $\mathcal{A}$ is an artinian and noetherian $\Bbbk$-linear abelian category in which the endomorphism ring of every simple object is $\Bbbk$;
(ii) E and F are endofunctors of $\mathcal{A}$ which are both left and right adjoint to each other;
(iii) mapping $\mathbf{e} \mapsto [\mathrm{E}]$ and $\mathbf{f} \mapsto [\mathrm{F}]$ extends to a locally finite $\mathfrak{sl}_2$-representation on $\mathbb{Q} \otimes_{\mathbb{Z}} [\mathcal{A}]$;
(iv) classes of simple objects in $\mathbb{Q} \otimes_{\mathbb{Z}} [\mathcal{A}]$ are eigenvectors for H;
(v) $\tau \in \mathrm{End}(E^2)$ is such that

$$(\mathrm{id}_E \circ_0 \tau) \circ_1 (\tau \circ_0 \mathrm{id}_E) \circ_1 (\mathrm{id}_E \circ_0 \tau) = (\tau \circ_0 \mathrm{id}_E) \circ_1 (\mathrm{id}_E \circ_0 \tau) \circ_1 (\tau \circ_0 \mathrm{id}_E);$$

(vi) $q \in \Bbbk^*$ is such that $(\tau + \mathrm{id}_{E^2})(\tau - q\,\mathrm{id}_{E^2}) = 0$;
(vii) $\chi \in \mathrm{End}(E)$ is such that

$$\tau \circ_1 (\mathrm{id}_E \circ_0 \chi) \circ_1 \tau = \begin{cases} q\,\chi \circ_0 \mathrm{id}_E, & q \neq 1; \\ \chi \circ_0 \mathrm{id}_E - \tau, & q = 1; \end{cases}$$

(viii) $a \in \Bbbk$ is such that $a \neq 0$ if $q \neq 1$ and $\chi - a$ is locally nilpotent.

We note that a can always be adjusted to either 0 or 1. Indeed, if $q \neq 1$ and $a \neq 0$, replacing χ by $a\chi$ gives a new $\mathfrak{sl}_2$-categorification with $a = 1$. If $q = 1$, replacing χ by $\chi + a\mathrm{id}_E$ gives a new $\mathfrak{sl}_2$-categorification with $a = 0$.

Example 13.2. Let $A_{-2} = A_2 := \Bbbk$ and $A_0 = \Bbbk[x]/(x^2)$ and set $\mathcal{A}_i := A_i$-mod for $i = 0, \pm 2$, $\mathcal{A} := \mathcal{A}_{-2} \oplus \mathcal{A}_0 \oplus \mathcal{A}_2$. Note that both A_{-2} and A_2 are unital subalgebras of A_0 and we have the corresponding induction and restriction functors Ind and Res. Hence we may define the functors E and F as given by the right and left arrows on the following picture, respectively:

$$\mathcal{A}_{-2} \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathcal{A}_0 \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} \mathcal{A}_2$$

Let $q = 1$ and $a = 0$. Define χ as multiplication by x on Res : $\mathcal{A}_0 \to \mathcal{A}_2$ and multiplication by $-x$ on Ind : $\mathcal{A}_{-2} \to \mathcal{A}_0$. Define $\tau \in \mathrm{End}_{\Bbbk}(\Bbbk[x]/(x^2))$ as the automorphism swapping 1 and x. It is easy to check that all conditions are satisfied and thus this is an $\mathfrak{sl}_2$-categorification (of the simple 3-dimensional $\mathfrak{sl}_2$-module).

Proposition 13.3. *Let* $(\mathcal{A}, \mathrm{E}, \mathrm{F}, q, a, \chi, \tau)$ *be an* $\mathfrak{sl}_2$*-categorification. For* $\lambda \in \mathbb{Z}$ *denote by* $\mathcal{A}_\lambda$ *the full subcategory of* $\mathcal{A}$ *consisting of all objects whose classes in the Grothendieck group have* $\mathfrak{sl}_2$*-weight* λ*. Then* $\mathcal{A} = \oplus_{\lambda \in \mathbb{Z}} \mathcal{A}_\lambda$.

13.2. **Affine Hecke algebras.** Let $n \in \mathbb{N}_0$ and $q \in \Bbbk^*$.

Definition 13.4 (Non-degenerate affine Hecke algebra)**.** If $q \neq 1$, then the affine Hecke algebra $\mathbb{H}_n(q)$ is defined as the $\Bbbk$-algebra with generators $T_1, \dots, T_{n-1}$, $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ subject to the following relations:

$$\begin{array}{rclrcll} (T_i+1)(T_i-q) & = & 0; & T_iT_j & = & T_jT_i, & |i-j|>1; \\ T_iT_{i+1}T_i & = & T_{i+1}T_iT_{i+1}; & X_iX_j & = & X_jX_i; & \\ X_iX_i^{-1} & = & 1; & X_i^{-1}X_i & = & 1; & \\ T_iX_iT_i & = & qX_{i+1}; & X_iT_j & = & T_jX_i, & i-j \neq 0,1. \end{array}$$

Note that the subalgebra $\mathbb{H}_n^f(q)$ of $\mathbb{H}_n(q)$, generated by the T_i's, is just an alternative version of the Hecke algebra $\mathbb{H}_n$ for $\mathbb{S}_n$ from Subsection 7.4.

Definition 13.5 (Degenerate affine Hecke algebra)**.** If $q = 1$, then the *degenerate* affine Hecke algebra $\mathbb{H}_n(1)$ is defined as the $\Bbbk$-algebra with generators $T_1, \dots, T_{n-1}$,

$X_1, \dots, X_n$ subject to the following relations:

$$\begin{array}{rclcrcll} T_i^2 & = & 1; & & T_iT_j & = & T_jT_i, & |i-j|>1; \\ T_iT_{i+1}T_i & = & T_{i+1}T_iT_{i+1}; & & X_iX_j & = & X_jX_i; & \\ X_{i+1}T_i & = & T_iX_i+1; & & X_iT_j & = & T_jX_i, & i-j\neq 0,1. \end{array}$$

Note that the degenerate affine Hecke algebra is not the specialization of the affine Hecke algebra at $q=1$. The subalgebra $\mathbb{H}_n^f(1)$ of $\mathbb{H}_n(1)$, generated by the T_i's, is isomorphic to the group algebra of the symmetric group. Relevance of affine Hecke algebras for sl$_2$-categorifications is justified by the following statement:

Lemma 13.6. *If $(\mathcal{A}, \mathrm{E}, \mathrm{F}, q, a, \chi, \tau)$ is an $\mathfrak{sl}_2$-categorification, then for all $n \in \mathbb{N}_0$ mapping*

$$T_i \mapsto \mathrm{id}_{E^{n-i-1}} \circ_0 \tau \circ_0 \mathrm{id}_{E^{i-1}} \qquad \text{and} \qquad X_i \mapsto \mathrm{id}_{E^{n-i}} \circ_0 \chi \circ_0 \mathrm{id}_{E^{i-1}}$$

extends to a homomorphism $\gamma_n : \mathbb{H}_n(q) \to \mathrm{End}(\mathrm{E}^n)$.

For $w \in W$ choose some reduces decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ and set $T_w := T_{i_1}T_{i_2}\cdots T_{i_k}$. Let $\mathtt{1}$ and sign denote the one-dimensional representations of $\mathbb{H}_n^f(q)$ given by $T_i \mapsto q$ and $T_i \mapsto -1$, respectively. Define

$$c_n^{\mathtt{1}} := \sum_{w\in W} T_w; \qquad\qquad c_n^{\mathrm{sign}} := \sum_{w\in W} (-q)^{-\mathfrak{l}(w)} T_w.$$

Then both $c_n^{\mathtt{1}}$ and c_n^{sign} belong to the center of $\mathbb{H}_n^f(q)$. For $\alpha \in \{\mathtt{1}, \mathrm{sign}\}$ and $n \in \mathbb{N}_0$ define $\mathrm{E}^{(\alpha,n)}$ to be the image of $c_n^\alpha : \mathrm{E}^n \to \mathrm{E}^n$.

Corollary 13.7. *For $\alpha \in \{\mathtt{1}, \mathrm{sign}\}$ and $n \in \mathbb{N}_0$ we have*

$$\mathrm{E}^n \cong \underbrace{\mathrm{E}^{(\alpha,n)} \oplus \mathrm{E}^{(\alpha,n)} \oplus \cdots \oplus \mathrm{E}^{(\alpha,n)}}_{n!\ \textit{summands}}.$$

Example 13.8. Let $A_{-2} = A_2 := \Bbbk[x]/(x^2)$ and $A_0 = \Bbbk$ and set $\mathcal{A}_i := A_i$-mod for $i = 0, \pm 2$, $\mathcal{A} := \mathcal{A}_{-2} \oplus \mathcal{A}_0 \oplus \mathcal{A}_2$. Then A_0 is a unital subalgebra of both A_{-2} and A_2 and we have the corresponding induction and restriction functors Ind and Res as in Example 13.2. Hence we may define the functors E and F as given by the right and left arrows on the following picture, respectively:

$$\mathcal{A}_{-2} \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} \mathcal{A}_0 \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathcal{A}_2$$

Note that this is a weak categorification of the simple 3-dimensional $\mathfrak{sl}_2$-module. The functor $\mathrm{E}^2 : \mathcal{A}_{-2} \to \mathcal{A}_2$ is isomorphic to tensoring with the bimodule $\Bbbk[x]/(x^2) \otimes_\Bbbk \Bbbk[x]/(x^2)$, which is easily seen to be indecomposable. This contradicts Corollary 13.7 and hence this is not an $\mathfrak{sl}_2$-categorification.

13.3. **Morphisms of $\mathfrak{sl}_2$-categorifications.** Let $\mathfrak{A} := (\mathcal{A}, \mathrm{E}, \mathrm{F}, q, a, \chi, \tau)$ and $\mathfrak{A}' := (\mathcal{A}', \mathrm{E}', \mathrm{F}', q', a', \chi', \tau')$ be two $\mathfrak{sl}_2$-categorifications. Denote by $\varepsilon : \mathrm{E}\circ\mathrm{F} \to \mathrm{Id}$, $\varepsilon' : \mathrm{E}'\circ\mathrm{F}' \to \mathrm{Id}$, $\eta : \mathrm{Id} \to \mathrm{E}\circ\mathrm{F}$ and $\eta' : \mathrm{Id} \to \mathrm{E}'\circ\mathrm{F}'$ some fixed counits and units, respectively.

Definition 13.9 ([CR])**.** A morphism of $\mathfrak{sl}_2$-categorifications from $\mathfrak{A}'$ to $\mathfrak{A}$ is a tuple (R, ξ, ζ) such that:

(i) $\mathrm{R} : \mathcal{A}' \to \mathcal{A}$ is a functor;
(ii) $\xi : \mathrm{R}\circ\mathrm{E}' \cong \mathrm{E}\circ\mathrm{R}$ is an isomorphism of functors;
(iii) $\zeta : \mathrm{R}\circ\mathrm{F}' \cong \mathrm{F}\circ\mathrm{R}$ is an isomorphism of functors;

(iv) the following diagram commutes:

$$\begin{array}{ccc} \mathrm{R}\circ\mathrm{F}' & \xrightarrow{\zeta} & \mathrm{F}\circ\mathrm{R} \\ {\scriptstyle\eta_{\mathrm{R}\circ\mathrm{F}'}}\downarrow & & \downarrow{\scriptstyle\mathrm{F}\circ\mathrm{R}(\varepsilon')} \\ \mathrm{F}\circ\mathrm{E}\circ\mathrm{R}\circ\mathrm{F}' & \xrightarrow{\mathrm{F}(\xi_{\mathrm{F}'}^{-1})} & \mathrm{F}\circ\mathrm{R}\circ\mathrm{E}'\circ\mathrm{F}'; \end{array}$$

(v) $a = a'$, $q = q'$ and the following diagrams commute:

$$\begin{array}{ccc} \mathrm{R}\circ\mathrm{E}' & \xrightarrow{\xi} & \mathrm{E}\circ\mathrm{R} \\ {\scriptstyle\mathrm{R}(\chi')}\downarrow & & \downarrow{\scriptstyle\chi_{\mathrm{R}}} \\ \mathrm{R}\circ\mathrm{E}' & \xrightarrow{\xi} & \mathrm{E}\circ\mathrm{R}; \end{array}$$

$$\begin{array}{ccccc} \mathrm{R}\circ\mathrm{E}'\circ\mathrm{E}' & \xrightarrow{\xi_{\mathrm{E}'}} & \mathrm{E}\circ\mathrm{R}\circ\mathrm{E}' & \xrightarrow{\mathrm{E}(\xi)} & \mathrm{E}\circ\mathrm{E}\circ\mathrm{R} \\ {\scriptstyle\mathrm{R}(\tau')}\downarrow & & & & \downarrow{\scriptstyle\tau_{\mathrm{R}}} \\ \mathrm{R}\circ\mathrm{E}'\circ\mathrm{E}' & \xrightarrow{\xi_{\mathrm{E}'}} & \mathrm{E}\circ\mathrm{R}\circ\mathrm{E}' & \xrightarrow{\mathrm{E}(\xi)} & \mathrm{E}\circ\mathrm{E}\circ\mathrm{R}. \end{array}$$

In the terminology from [Kh], a morphism of two $\mathfrak{sl}_2$-categorifications is a functor from $\mathcal{A}'$ to $\mathcal{A}$ which *naturally* intertwines the functorial action of $\mathfrak{sl}_2$ on these two categories (see Subsection 3.8).

13.4. **Minimal $\mathfrak{sl}_2$-categorification of simple finite dimensional modules.** Let a and q be as above. Let $\mathbb{P}_n$ denote the subalgebra of $\mathbb{H}_n(q)$, generated by $X_1,\dots,X_n$ (and $X_1^{-1},\dots,X_n^{-1}$ if $q\neq 1$). Denote by $\mathfrak{m}_n$ the ideal of $\mathbb{P}_n$ generated by $X_i - a$ and set $\mathfrak{n}_n = (\mathfrak{m}_n)^{\mathbb{S}_n}$ (were $\mathbb{S}_n$ acts on $\mathbb{P}_n$, as usual, by permuting the variables). Set $\overline{\mathbb{H}}_n(q) := \mathbb{H}_n(q)/(\mathbb{H}_n(q)\mathfrak{n}_n)$.

For $k\leq n$ there is the obvious natural map $\mathbb{H}_k(q)\mapsto\mathbb{H}_n(q)$ (it is the identity on generators of $\mathbb{H}_k(q)$). We denote by $\overline{\mathbb{H}}_{k,n}(q)$ the image of $\mathbb{H}_k$ in $\overline{\mathbb{H}}_n(q)$. One can show that the algebra $\overline{\mathbb{H}}_{k,n}(q)$ is local, symmetric and independent of a and q up to isomorphism.

Lemma 13.10. *For $i\leq j$ the algebra $\overline{\mathbb{H}}_{j,n}(q)$ is a free $\overline{\mathbb{H}}_{i,n}(q)$-module of rank $\frac{(n-i)!j!}{(n-j)!i!}$.*

For a fixed $n\in\mathbb{N}_0$ set $B_i := \overline{\mathbb{H}}_{i,n}(q)$, $i = 0,1,2\dots,n$. Define the category $\mathcal{A}(n)$ as follows:

$$\mathcal{A}(n) := \bigoplus_{i=0}^{n}\mathcal{A}(n)_{-n+2i}, \qquad \text{where} \qquad \mathcal{A}(n)_{-n+2i} := B_i\text{-mod}.$$

Set further

$$\mathrm{E} := \bigoplus_{i=0}^{n-1}\mathrm{Ind}_{B_i}^{B_{i+1}}, \qquad \mathrm{F} := \bigoplus_{i=1}^{n}\mathrm{Res}_{B_{i-1}}^{B_i}.$$

The image of X_{i+1} in B_{i+1} gives an endomorphism of $\mathrm{Ind}_{B_i}^{B_{i+1}}$ by the right multiplication. Taking the sum over all i, we get an endomorphism of E, which we denote by χ. Similarly, the image of T_{i+1} in B_{i+2} gives an endomorphism of $\mathrm{Ind}_{B_i}^{B_{i+2}}$ and, taking the sum over all i, we get an endomorphism of E^2, which we denote by τ. The second part of the following statement from [CR] is truly remarkable:

Theorem 13.11 (Minimal categorification and its uniqueness)**.** (*a*) *The tuple $\mathfrak{A} := (\mathcal{A}(n),\mathrm{E},\mathrm{F},q,a,\chi,\tau)$ from the above is an $\mathfrak{sl}_2$-categorification of the simple $n+1$-dimensional $\mathfrak{sl}_2$-module.*

(b) *If* $\mathfrak{A}' := (\mathcal{A}', \mathrm{E}', \mathrm{F}', q, a, \chi', \tau')$ *is an* $\mathfrak{sl}_2$*-categorification of the simple* $n+1$*-dimensional* $\mathfrak{sl}_2$*-module such that the (unique) simple object annihilated by* F' *is projective, then the* $\mathfrak{sl}_2$*-categorifications* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are equivalent.*

13.5. **$\mathfrak{sl}_2$-categorification on category $\mathcal{O}$.** In this subsection we assume $q = 1$. Let $a \in \mathbb{C}$. For $\lambda, \mu \in \mathfrak{h}^*$ write $\lambda \to_a \mu$ provided that there exists $j \in \{1, 2, \dots, n\}$ such that $\lambda_j - j + 1 = a - 1$, $\mu_j - j + 1 = a$ and $\lambda_i = \mu_i$ for all $i \neq j$. For $\lambda, \mu \in \mathfrak{h}^*_{\mathrm{dom}}$ we write $\chi_\lambda \to_a \chi_\mu$ provided that there exist $\lambda' \in W \cdot \lambda$ and $\mu' \in W \cdot \mu$ such that $\lambda' \to_a \mu'$.

Let V be the natural n-dimensional representation of $\mathfrak{g}$. Then the projective endofunctor $V \otimes_{\mathbb{C}} {}_-$ of $\mathcal{O}$ decomposes as follows:

$$V \otimes_{\mathbb{C}} {}_- = \bigoplus_{a \in \mathbb{C}} \mathrm{E}_a, \qquad \text{where} \qquad \mathrm{E}_a = \bigoplus_{\lambda, \mu \in \mathfrak{h}^*_{\mathrm{dom}}; \chi_\lambda \to_a \chi_\mu} \mathrm{Pr}_{\mathcal{O}_\mu} \circ (V \otimes_{\mathbb{C}} {}_-) \circ \mathrm{Pr}_{\mathcal{O}_\lambda},$$

here the projection Pr is defined with respect to the decomposition from Theorem 4.3. Denote by F_a the (both left and right) adjoint of E_a.

Define $\tau \in \mathrm{End}((V \otimes_{\mathbb{C}} {}_-)^2)$ as follows: for $M \in \mathcal{O}$, $m \in M$ and $v, v' \in V$ set $\tau_M(v \otimes v' \otimes m) := v' \otimes v \otimes m$. Consider the element $\Omega := \sum_{i,j=1}^n e_{ij} \otimes e_{ji} \in \mathfrak{g} \otimes \mathfrak{g}$. Define $\chi \in \mathrm{End}(V \otimes_{\mathbb{C}} {}_-)$ as follows: for $M \in \mathcal{O}$, $m \in M$ and $v \in V$ set $\chi_M(v \otimes m) := \Omega(v \otimes m)$.

Lemma 13.12. *For every* $a \in \mathbb{C}$ *the natural transformations* χ *and* τ *restrict to* E_a *and* E_a^2*, respectively.*

We denote by χ_a and τ_a the restrictions of χ and τ to E_a and E_a^2, respectively.

Proposition 13.13 ([CR])**.** *For every* $a \in \mathbb{C}$ *the tuple* $(\mathcal{O}, \mathrm{E}_a, \mathrm{F}_a, 1, a, \chi_a, \tau_a)$ *is an* $\mathfrak{sl}_2$*-categorification.*

Corollary 13.14. *The categorification of the simple* $n+1$*-dimensional* $\mathfrak{sl}_2$*-module from Subsection 11.4 is an* $\mathfrak{sl}_2$*-categorification.*

13.6. **Categorification of the simple reflection.** Consider the element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp(-\mathbf{f})\exp(\mathbf{e})\exp(-\mathbf{f}) \in \mathrm{SL}_2.$$

Lemma 13.15. *Let* V *be an integrable* $\mathfrak{sl}_2$*-module, that is a direct sum of finite dimensional* $\mathfrak{sl}_2$*-modules. Let* $\lambda \in \mathbb{Z}$ *and* $v \in V_\lambda$*.*

(a) *The action of* S *induces an isomorphism between* V_λ *and* $V_{-\lambda}$*.*

(b)

$$S(v) = \sum_{i=\max(0,-\lambda)}^{\max\{j : \mathbf{f}^j(v) \neq 0\}} \frac{(-1)^i}{i!(\lambda+i)!} \mathbf{e}^{\lambda+i} \mathbf{f}^i(v).$$

Consider some $\mathfrak{sl}_2$-categorification $(\mathcal{A}, \mathrm{E}, \mathrm{F}, q, a, \chi, \tau)$. Let $\lambda \in \mathbb{Z}$. For $i \in \mathbb{Z}$ such that $i, \lambda + i \geq 0$, denote by $(\Theta_\lambda)^{-i}$ the restriction of $\mathrm{E}^{(\mathrm{sign}, \lambda+i)} \circ \mathrm{F}^{(1,i)}$ to $\mathcal{A}_\lambda$. Set $(\Theta)^{-i} = 0$ otherwise. The map

$$f : \mathrm{E}^{\lambda+i} \circ \mathrm{F}^i = \mathrm{E}^{\lambda+i-1} \circ \mathrm{E} \circ \mathrm{F} \circ \mathrm{F}^{i-1} \xrightarrow{\mathrm{id}_{\mathrm{E}^{\lambda+i-1}} \circ_0 \varepsilon \circ_0 \mathrm{id}_{\mathrm{F}^{i-1}}} \mathrm{E}^{\lambda+i-1} \circ \mathrm{F}^{i-1}$$

restricts to a map

$$d^{-i} : \mathrm{E}^{(\mathrm{sign}, \lambda+i)} \circ \mathrm{F}^{(1,i)} \to \mathrm{E}^{(\mathrm{sign}, \lambda+i-1)} \circ \mathrm{F}^{(1,i-1)}.$$

Set

$$\Theta_\lambda := \quad \cdots \xrightarrow{d^{-i-1}} (\Theta_\lambda)^{-i} \xrightarrow{d^{-i}} (\Theta_\lambda)^{-i+1} \xrightarrow{d^{-i+1}} \cdots$$

and put $\Theta := \oplus_\lambda \Theta_\lambda$. The following is described by the authors of [CR] as their main result:

Theorem 13.16 ([CR]). *We have the following:*
(*a*) Θ_λ *is a complex.*
(*b*) *The action of* $[\Theta_\lambda] : [\mathcal{A}_\lambda] \to [\mathcal{A}_{-\lambda}]$ *coincides with the action of* S.
(*c*) *The complex of functors* Θ *induces a self-equivalence of* $\mathcal{K}^b(\mathcal{A})$ *and of* $\mathcal{D}^b(\mathcal{A})$ *and restricts to equivalences* $\mathcal{K}^b(\mathcal{A}_\lambda) \tilde{\to} \mathcal{K}^b(\mathcal{A}_{-\lambda})$ *and* $\mathcal{D}^b(\mathcal{A}_\lambda) \tilde{\to} \mathcal{D}^b(\mathcal{A}_{-\lambda})$.

Idea of the proof. First, prove the claim for a minimal categorification. Use the first step to prove the claim for any categorification of an isotypic module. Finally, there is a filtration of $\mathcal{A}$ by Serre subcategories such that subquotients of this filtration descend exactly to isotypic $\mathfrak{sl}_2$-components in $[\mathcal{A}]$. Deduce now the claim in the general case using this filtration and step two. □

For a construction of a 2-category categorifying $U(\mathfrak{sl}_2)$ see [La, Ro2].

14. Application: blocks of $\mathbb{F}[\mathbb{S}_n]$ and Broué's conjecture

14.1. **Jucys-Murphy elements and formal characters.** In the first part of this section we closely follow the exposition from [BrKl]. Let $\mathbb{F}$ denote a field of arbitrary characteristic p and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. For $n \in \mathbb{N}_0$ consider the symmetric group $\mathbb{S}_n$.

Definition 14.1. For $k = 1, \dots, n$ define the *Jucys-Murphy* element

$$\mathtt{x}_k := (1, k) + (2, k) + \cdots + (k-1, k) \in \mathbb{F}[\mathbb{S}_n].$$

Importance of Jucys-Murphy elements is explained by the following:

Proposition 14.2 ([Ju, Mu]). *The elements* $\mathtt{x}_1$, $\mathtt{x}_2, \dots,$ $\mathtt{x}_n$ *commute with one other and the center of the group algebra* $\mathbb{F}[\mathbb{S}_n]$ *is precisely the set of symmetric polynomials in these elements.*

Let M be a finite dimensional $\mathbb{F}[\mathbb{S}_n]$-module. For $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathbb{F}_p^n$ set

$$M_{\mathbf{i}} := \{v \in M : (\mathtt{x}_r - i_r)^N v = 0 \text{ for all } N \gg 0 \text{ and } r = 1, 2, \dots, n\}.$$

Amazingly enough, we have the following property:

Proposition 14.3 ([BrKl]). *We have* $M = \bigoplus_{\mathbf{i} \in \mathbb{F}_p^n} M_{\mathbf{i}}$.

Consider the free $\mathbb{Z}$-module Char with basis $\{\exp(\mathbf{i}) : \mathbf{i} \in \mathbb{F}_p^n\}$.

Definition 14.4. Let M be an $\mathbb{F}[\mathbb{S}_n]$-module. The *formal character* of M is

$$\operatorname{ch} M := \sum_{\mathbf{i} \in \mathbb{F}_p^n} \dim M_{\mathbf{i}} \exp(\mathbf{i}) \in \text{Char}.$$

Proposition 14.5 ([Va]). (*a*) *If* $M \hookrightarrow N \twoheadrightarrow K$ *is a short exact sequence of* $\mathbb{F}[\mathbb{S}_n]$-*modules, then* $\operatorname{ch} N = \operatorname{ch} M + \operatorname{ch} K$.
(*b*) *The formal characters of the inequivalent irreducible* $\mathbb{F}[\mathbb{S}_n]$-*modules are linearly independent.*

Define the set

$$\Gamma_n := \{\gamma = (\gamma_r)_{r \in \mathbb{F}_p} : \gamma_r \in \mathbb{N}_0, \sum_{r \in \mathbb{F}_p} \gamma_r = n\}.$$

For $\mathbf{i} \in \mathbb{F}_p^n$ set

$$\operatorname{wt}(\mathbf{i}) := (\gamma_r)_{r \in \mathbb{F}_p} \in \Gamma_n, \quad \text{where } \gamma_r = |\{j \in \{1, 2, \dots, n\} : i_j = r\}|.$$

For $\gamma \in \Gamma_n$ set

$$M(\gamma) := \bigoplus_{\mathbf{i} \in \mathbb{F}_p^n : \operatorname{wt}(\mathbf{i}) = \gamma} M_{\mathbf{i}} \subset M.$$

Corollary 14.6. *For every $\gamma \in \Gamma_n$ the subspace $M(\gamma)$ is a submodule of M.*

As a consequence, we have the following decomposition of $\mathbb{F}[\mathbb{S}_n]$-mod into *blocks*:

$$\mathbb{F}[\mathbb{S}_n]\text{-mod} \cong \bigoplus_{\gamma\in\Gamma_n} \mathcal{B}_\gamma,$$

where $\mathcal{B}_\gamma$ denotes the full subcategory of $\mathbb{F}[\mathbb{S}_n]$-mod consisting of all modules M satisfying $M = M(\gamma)$.

14.2. **Induction and restriction.** For every $i \in \mathbb{F}_p$ define the insertion operation $\mathrm{ins}_i : \Gamma_n \to \Gamma_{n+1}$ as follows: for $\gamma = (\gamma_r)_{r\in\mathbb{F}_p}$ set $\mathrm{ins}_i(\gamma)$ to be $(\gamma'_r)_{r\in\mathbb{F}_p}$ such that $\gamma'_r = \gamma_r$ for all $r \neq i$ and $\gamma'_i = \gamma_i + 1$.

For every $i \in \mathbb{F}_p$ define the removal operation $\mathrm{rem}_i : \Gamma_n \to \Gamma_{n-1}$ (this will be a partially defined map) as follows: for $\gamma = (\gamma_r)_{r\in\mathbb{F}_p}$ the image of γ under rem_i is defined if and only if $\gamma_i > 0$, and, if this condition is satisfied, we set $\mathrm{rem}_i(\gamma)$ to be $(\gamma'_r)_{r\in\mathbb{F}_p}$ such that $\gamma'_r = \gamma_r$ for all $r \neq i$ and $\gamma'_i = \gamma_i - 1$.

For every $i \in \mathbb{F}_p$ we now define functors

$$\mathrm{E}_i : \mathbb{F}[\mathbb{S}_n]\text{-mod} \to \mathbb{F}[\mathbb{S}_{n-1}]\text{-mod} \quad \text{and} \quad \mathrm{F}_i : \mathbb{F}[\mathbb{S}_n]\text{-mod} \to \mathbb{F}[\mathbb{S}_{n+1}]\text{-mod}$$

as follows: For $\gamma \in \Gamma_n$ and $M \in \mathcal{B}_\gamma$ set

$$\mathrm{E}_i\, M := \begin{cases} (\mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} M)(\mathrm{rem}_i(\gamma)), & \mathrm{rem}_i(\gamma) \text{ is defined;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{F}_i\, M := (\mathrm{Ind}^{\mathbb{S}_{n+1}}_{\mathbb{S}_n} M)(\mathrm{ins}_i(\gamma)).$$

Extending additively, this defines E_i and F_i on the whole of $\mathbb{F}[\mathbb{S}_n]$-mod. Directly from the definition we obtain the following: for every $\mathbb{F}[\mathbb{S}_n]$-module M we have

$$\mathrm{Res}^{\mathbb{S}_n}_{\mathbb{S}_{n-1}} M \cong \bigoplus_{i\in\mathbb{F}_p} \mathrm{E}_i\, M, \qquad \mathrm{Ind}^{\mathbb{S}_{n+1}}_{\mathbb{S}_n} M \cong \bigoplus_{i\in\mathbb{F}_p} \mathrm{F}_i\, M.$$

Using the Frobenius reciprocity, it follows that E_i is both left and right adjoint of F_i. In particular, both E_i and F_i are exact functors.

14.3. **Categorification of the basic representation of an affine Kac-Moody algebra.** Denote by $\mathtt{R}_n$ the free $\mathbb{Z}$-module spanned by the formal characters of irreducible $\mathbb{F}[\mathbb{S}_n]$-modules, the so-called *character ring* of $\mathbb{F}[\mathbb{S}_n]$. By Proposition 14.5(b), the map ch induces an isomorphism between $\mathtt{R}_n$ and the Grothendieck group of the category of all finite dimensional $\mathbb{F}[\mathbb{S}_n]$-modules. Set

$$\mathtt{R} := \bigoplus_{n\in\mathbb{N}_0} \mathtt{R}_n.$$

The exact functors E_i and F_i induce $\mathbb{Z}$-linear endomorphisms of $\mathtt{R}$. Extending the scalars to $\mathbb{C}$ we get the vector space

$$\mathtt{R}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Z} \mathtt{R}$$

and linear operators $[\mathrm{E}_i]$ and $[\mathrm{F}_i]$ on it.

Theorem 14.7 ([LLT, Ari, Gr]). (*a*) *The linear operators $[\mathrm{E}_i]$ and $[\mathrm{F}_i]$, $i \in \mathbb{F}_p$, satisfy the defining relations for the Chevalley generators of an affine Kac-Moody Lie algebra $\mathfrak{a}$ of type $A^{(1)}_{p-1}$ (resp. A_∞ in the case $p = 0$).*

(*b*) *The $\mathfrak{a}$-module $\mathtt{R}_\mathbb{C}$, given by* (a)*, is isomorphic to the basic highest weight representation $V(\Lambda_0)$ of this Kac-Moody algebra, where $\Lambda_0 = (1, 0, 0, \dots)$, and is generated by the highest weight vector, represented by the character of the irreducible $\mathbb{F}[\mathbb{S}_0]$-module.*

14.4. **Broué's conjecture.** In this overview we closely follow [Ric] and refer the reader to this nice survey for more details. Let G be a finite group and $\mathbb{F}$ be a field of characteristic $p > 0$. The finite dimensional group $\mathbb{F}$-algebra $\mathbb{F}[G]$ decomposes into a direct sum of indecomposable subalgebras, called *blocks*. Let A be a block of $\mathbb{F}[G]$. A *defect group* of A is a minimal subgroup D of G such that every A-module is a direct summand of a module induced from $\mathbb{F}[D]$. A defect group is always a p-subgroup of G and is determined uniquely up to conjugacy in G. Denote by $N_G(D)$ the normalizer of D in G.

Theorem 14.8 (Brauer's First Main Theorem). *If D is a p-subgroup of G, then there is a natural bijection between the blocks of $\mathbb{F}[G]$ with defect group D and the blocks of $\mathbb{F}[N_G(D)]$ with defect group D.*

If A is a block of G with defect group D, then the block B of $N_G(D)$ corresponding to A via the correspondence described by Theorem 14.8 is called the *Brauer correspondent* of A. For example, if A-mod contains the trivial G-module, then the defect group D of A is a Sylow p-subgroup of G and the Brauer correspondent of A is the block of $\mathbb{F}[N_G(D)]$ containing the trivial $N_G(D)$-module.

Conjecture 14.9 (Alperin's Weight Conjecture for abelian defect groups). Let G be a finite group, A a block of G with an abelian defect group D and B the Brauer correspondent of A. Then A-mod and B-mod have the same number of isomorphism classes of simple modules.

The number of isomorphism classes of simple modules can be interpreted as an isomorphism between the Grothendieck groups of A-mod and B-mod. Conceptually, it would be nice if this isomorphism would come from some "higher level" isomorphism, for example from an isomorphism (or Morita equivalence) of A and B. Unfortunately, small examples show that this cannot be expected. However, there is still a conjectural higher level isomorphism as described in:

Conjecture 14.10 (Broué's abelian defect group Conjecture). Let G be a finite group, A a block of G with an abelian defect group D and B the Brauer correspondent of A. Then $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

14.5. **Broué's conjecture for $\mathbb{S}_n$.** Let us go back to the character ring $\mathtt{R}_n$ of $\mathbb{F}[\mathbb{S}_n]$, whose complexification $\mathtt{R}_{\mathbb{C}}$ was realized in Subsection 14.3 as the basic highest weight representation $V(\Lambda_0)$ over some affine Kac-Moody Lie algebra $\mathfrak{a}$. It turns out that the theory of blocks has a natural description in terms of this categorification picture:

Theorem 14.11 ([LLT, CR]). (*a*) *The decomposition of $\mathtt{R}$ into blocks gives exactly the weight space decomposition of $V(\Lambda_0)$ with respect to the standard Cartan subalgebra of $\mathfrak{g}$.*

(*b*) *Two blocks of symmetric groups have isomorphic defect groups if and only if the corresponding weight spaces are in the same orbit under the action of the affine Weyl group.*

Now we can state the principal application of $\mathfrak{sl}_2$-categorification from [CR]:

Theorem 14.12 ([CR]). *Let $p > 0$ and $\mathbb{F}$ be either $\mathbb{Z}_{(p)}$ or an algebraically closed field of characteristic p.*

(*a*) *For every $i \in \mathbb{F}_p$ the action of E_i and F_i on*

$$\mathfrak{S} := \bigoplus_{n\in\mathbb{N}_0} \mathbb{F}[\mathbb{S}_n]\text{-mod}$$

extends to an $\mathfrak{sl}_2$-categorification over $\mathbb{F}$.

(*b*) *Let A and B be two blocks of symmetric groups over $\mathbb{F}$ with isomorphic defect groups. Then $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.*

Idea of the proof of (b). Because of Theorem 14.11(a), it is enough to show that two blocks which are connected by a simple reflection of the affine Weyl group are derived equivalent. To prove this, use (a) and Theorem 13.16. □

Corollary 14.13. *Broué's abelian defect group Conjecture holds for $G = \mathbb{S}_n$.*

14.6. **Divided powers.** Consider again the functorial action of the affine Kac-Moody Lie algebra $\mathfrak{a}$ on the direct sum of module categories of all symmetric groups as constructed in Subsection 14.3. There is an important set of elements of $\mathfrak{a}$ called *divided powers* and defined (in terms of the Chevalley generators e_i's and f_i's of $\mathfrak{a}$) as follows:

$$e_i^{(r)} := \frac{e_i^r}{r!} \qquad \text{and} \qquad f_i^{(r)} := \frac{f_i^r}{r!}.$$

It turns out that these elements also admit a natural functorial description. As in Subsection 14.2, we let $i \in \mathbb{F}_p$.

Let M be an $\mathbb{F}[\mathbb{S}_n]$-module and assume that there exists $\gamma \in \Gamma_n$ such that $M = M(\gamma)$. Fix $r \in \mathbb{N}$. View M as an $\mathbb{F}(\mathbb{S}_n \oplus \mathbb{S}_r)$-module by letting $\mathbb{S}_r$ act trivially. Embed $\mathbb{S}_n \oplus \mathbb{S}_r$ into $\mathbb{S}_{n+r}$ in the obvious way and define

$$\mathrm{F}_i^{(r)}\, M := (\mathrm{Ind}_{\mathbb{S}_n \oplus \mathbb{S}_r}^{\mathbb{S}_{n+r}} M)(\mathrm{ins}_i^r(\gamma)).$$

To define $\mathrm{E}_i^{(r)}$ write $n = k + r$. If $k < 0$, then the definition below gives $\mathrm{E}_i^{(r)}\, M = 0$. If $k \geq 0$, consider the obvious embedding of $\mathbb{S}_k \oplus \mathbb{S}_r$ into $\mathbb{S}_n$. Then the set $M^{\mathbb{S}_r}$ of $\mathbb{S}_r$-fixed points in M is naturally an $\mathbb{S}_k$-module. Set

$$\mathrm{E}_i^{(r)}\, M := \begin{cases} (M^{\mathbb{S}_r})(\mathrm{rem}_i^r(\gamma)), & \mathrm{rem}_i^r(\gamma) \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases}$$

Extending additively, defines functors $\mathrm{E}_i^{(r)}$ and $\mathrm{F}_i^{(r)}$ on all $\mathbb{F}[\mathbb{S}_n]$-modules and thus gives the corresponding endofunctors of $\mathfrak{S}$. Clearly, $\mathrm{E}_i^{(r)}$ is both left and right adjoint to $\mathrm{F}_i^{(r)}$. Denote by $\mathtt{R}_n^*$ the $\mathbb{Z}$-submodule of $\mathtt{R}_n$ spanned by the formal characters of indecomposable projective $\mathbb{F}[\mathbb{S}_n]$-modules and set

$$\mathtt{R}^* := \bigoplus_{n \in \mathbb{N}_0} \mathtt{R}_n^*.$$

Theorem 14.14 ([Gr])**.** (*a*) *There are isomorphisms of functors as follows:*

$$\mathrm{E}_i^r \cong (\mathrm{E}_i^{(r)})^{\oplus r!} \qquad \textit{and} \qquad \mathrm{F}_i^r \cong (\mathrm{F}_i^{(r)})^{\oplus r!}.$$

(*b*) *The lattice $\mathtt{R}^* \subset \mathtt{R}_{\mathbb{C}}$ is the $\mathbb{Z}$-submodule of $\mathtt{R}_{\mathbb{C}}$ generated by the character of the irreducible $\mathbb{F}[\mathbb{S}_0]$-module under the action of $\mathrm{F}_i^{(r)}$, $i \in \mathbb{F}_p$, $r \in \mathbb{N}_0$.*

(*c*) *The lattice $\mathtt{R} \subset \mathtt{R}_{\mathbb{C}}$ is dual to $\mathtt{R}^*$ with respect to the Shapovalov form.*

Claim (a) of Theorem 14.14 can be derived from Corollary 13.7 and Theorem 14.12(a).

15. Applications: of $\mathbb{S}_n$-categorifications

15.1. **Wedderburn basis for $\mathbb{C}[\mathbb{S}_n]$.** Artin-Wedderburn theorem says that every semi-simple complex finite dimensional algebra A is isomorphic to a direct sum of matrix algebras of the form $\mathrm{Mat}_{k\times k}(\mathbb{C})$. If such a decomposition of A is fixed, then the basis of A, consisting of matrix units in all components, is called a *Wedderburn basis* for A. A nice property of this basis is that if we take the linear span of basis elements along a fixed row in some matrix component $\mathrm{Mat}_{k\times k}(\mathbb{C})$ of A, then this linear span is stable with respect to the right multiplication with elements from

A and thus gives a simple submodule of the right regular module A_A. In other words, each Wedderburn basis for A automatically gives a decomposition of the right regular A-module into a direct sum of simple modules. For the left regular module one should just consider columns in matrix components instead of rows.

Consider $A = \mathbb{C}[\mathbb{S}_n]$, which is a semi-simple algebra by Maschke's Theorem. The right regular representation of A is categorified in Proposition 5.7 via the action of projective functors on a regular integral block $\mathcal{O}_\lambda$ for the algebra $\mathfrak{gl}_n$. In this categorification picture the standard basis of the module $\mathbb{C}[\mathbb{S}_n]$ is given by the classes $[\Delta(w \cdot \lambda)]$ of Verma modules in $\mathcal{O}_\lambda$. For a simple reflection s, the element $e + s$ of $\mathbb{C}[\mathbb{S}_n]$ acts on $\mathcal{O}_\lambda$ via the projective functor θ_s.

For $w \in \mathbb{S}_n$ denote by $\overline{w}$ the unique involution in the right cell of w, that is the element of $\mathbb{S}_n$ such that $p(\overline{w}) = q(\overline{w}) = p(w)$, where $(p(w), q(w))$ is the pair of standard Young tableaux associated to w by the Robinson-Schensted correspondence (see [Sa, Section 3.1]). Consider the *evaluation map*

$$\mathrm{ev} : \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{H} \overset{\mathrm{proj}}{\longrightarrow} (\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{H})/(v-1) \overset{\sim}{\longrightarrow} \mathbb{C}[\mathbb{S}_n],$$

which sends $1 \otimes H_w$ to w. For $w \in \mathbb{S}_n$ define

$$f_w := \mathrm{ev}(\underline{\hat{H}}_{\overline{w}} \underline{H}_w).$$

Example 15.1. In the case $n = 3$ denote by s and t the simple reflections in $\mathbb{S}_3$. Then we have:

$$\begin{aligned}
f_e &= (e - s - t + st + ts - sts)e = e - s - t + st + ts - sts;\\
f_s &= (s - st - ts + sts)(e + s) = e + s - t - ts;\\
f_s &= (t - st - ts + sts)(e + t) = e + t - s - st;\\
f_{st} &= (s - st - ts + sts)(e + s + t + st) = s + st - ts - sts;\\
f_{ts} &= (t - st - ts + sts)(e + s + t + ts) = t + ts - st - sts;\\
f_{sts} &= sts(e + s + t + ts + st + sts) = e + s + t + ts + st + sts.
\end{aligned}$$

Now observe that, by Proposition 5.7, the element f_w can be interpreted as $[\theta_w L(\overline{w} \cdot \lambda)]$.

Proposition 15.2 ([MS5])**.** *Let $w \in \mathbb{S}_n$ and $\mathcal{R}$ be the right cell of w. Then the module $\theta_w L(\overline{w} \cdot \lambda)$ is indecomposable and both projective and injective in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$.*

Combining Propositions 15.2 and 9.2 we see that $\{\theta_x L(\overline{w} \cdot \lambda) : x \in \mathcal{R}\}$ is a complete and irredundant list of indecomposable projective-injective modules in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$. By Theorem 9.3, the action of projective functors preserves the additive category of projective-injective modules in $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}$ and, going to the Grothendieck group, categorifies the Kazhdan-Lusztig cell module of $\mathcal{R}$. For $\mathbb{S}_n$ we know that all cell modules are irreducible. This implies the following statement:

Theorem 15.3 ([MS5])**.** (*a*) *The elements $\{f_w : w \in \mathbb{S}_n\}$ form a basis of $\mathbb{C}[\mathbb{S}_n]$.*
(*b*) *For $w \in \mathbb{S}_n$ let $\mathbf{S}(w)$ denote the linear span of f_x, where x is in the right cell of w. Then $\mathbf{S}(w)$ is an irreducible submodule of the right regular module $\mathbb{C}[\mathbb{S}_n]$.*

We also have the following statement:

Proposition 15.4 ([Ne])**.** *The basis f_w is, up to normalization, a Wedderburn basis for $\mathbb{C}[\mathbb{S}_n]$.*

The normalization factors which appear in Proposition 15.4 can be formulated in terms of the dimension of the endomorphism algebra of the modules $\theta_x L(\overline{w} \cdot \lambda)$. This gives a very clear categorical interpretation of this Wedderburn basis for $\mathbb{C}[\mathbb{S}_n]$.

15.2. **Kostant's problem.** For every two $\mathfrak{g}$-modules M and N the space $\mathrm{Hom}_{\mathbb{C}}(M,N)$ carries the natural structure of a $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule. Denote by $\mathcal{L}(M,N)$ the subspace of $\mathrm{Hom}_{\mathbb{C}}(M,N)$ consisting of all elements, the adjoint action of $\mathfrak{g}$ on which is locally finite. The space $\mathcal{L}(M,N)$ turns out to be a subbimodule of $\mathrm{Hom}_{\mathbb{C}}(M,N)$, see [Ja2, Kapitel 6].

Since the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ is locally finite, for any $\mathfrak{g}$-module M the natural image of $U(\mathfrak{g})$ in $\mathrm{Hom}_{\mathbb{C}}(M,M)$ belongs to $\mathcal{L}(M,M)$. The kernel of this map is, by definition, the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(M)$ of M, which gives us the following injective map of $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules:

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(M) \hookrightarrow \mathcal{L}(M,M). \tag{15.1}$$

The classical problem of Kostant, as described in [Jo3], can be formulated in the following way:

Problem 15.5 (Kostant's problem)**.** *For which $\mathfrak{g}$-modules M the map* (15.1) *is surjective (and hence an isomorphism)?*

Unfortunately, the complete answer to Kostant's problem in not even known for simple highest weight modules. However, many special cases are settled. Here is a list of known results:

- Kostant's problem has positive answer for all Verma modules, see [Jo3, Ja2].
- Kostant's problem has positive answer for all quotients of $\Delta(\lambda)$ if $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$, see [Ja2, 6.9].
- If $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and regular and $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra, then Kostant's problem has positive answer for $L(w_o^{\mathfrak{p}} w_o \cdot \lambda)$, see [GJ].
- If $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and regular and $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra, then Kostant's problem has positive answer for all quotients of $\Delta(w_o^{\mathfrak{p}} w_o \cdot \lambda)$, see [Kaa].
- If $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and regular, $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra and $s \in W_{\mathfrak{p}}$ is a simple reflection, then Kostant's problem has positive answer for $L(s w_o^{\mathfrak{p}} w_o \cdot \lambda)$, see [Ma1].
- If $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and regular, $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ are parabolic subalgebras, then Kostant's problem has positive answer for $L(w_o^{\mathfrak{q}} w_o^{\mathfrak{p}} w_o \cdot \lambda)$, see [Kaa].
- If $\mathfrak{g}$ is of type B_2 with two simple reflections s and t, and $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and regular, then Kostant's problem has *negative* answer for $L(st \cdot \lambda)$, see [Jo3].
- If $\mathfrak{g}$ is of type A_3 with three simple reflections r, s and t (such that $rt = tr$), and $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ is integral and regular, then Kostant's problem has *negative* answer for $L(rt \cdot \lambda)$, see [MS5].

Some further results on Kostant's problem can be found in [KaMa, Ma3, MS4, MM1] (see also references therein). In particular, in [Ma3] it is shown that the positive answer to Kostant's problem for certain simple highest weight modules can be equivalently reformulated in terms of the double centralizer property with respect to projective injective modules in the category $\mathcal{O}^{\hat{\mathcal{R}}}_{\lambda}$.

In this subsection we describe one of the main results from [MS4]. It is related to Kostant's problem and its proof is based on the idea of categorification. We have the following classical statement:

Theorem 15.6 ([Vo, Jo1])**.** *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be dominant and regular. Then for any $x, y \in \mathbb{S}_n$ the equality $\mathrm{Ann}_{U(\mathfrak{g})}(L(x \cdot \lambda)) = \mathrm{Ann}_{U(\mathfrak{g})}(L(y \cdot \lambda))$ is equivalent to the condition $x \sim_L y$.*

Theorem 15.6 described the left hand side of (15.1). In the case $M \in \mathcal{O}$, the $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule on the right hand side of (15.1) is a Harish-Chandra bimodule (see Subsection 5.3). In particular, with respect to the adjoint action of $\mathfrak{g}$, the bimodule $\mathcal{L}(M, M)$ decomposes into a direct sum of simple finite dimensional $\mathfrak{g}$-modules. Moreover, if V is a simple finite dimensional $\mathfrak{g}$-module, then, by [Ja2, 6.8], we have:

$$[\mathcal{L}(M, M) : V] = \dim \mathrm{Hom}_{\mathfrak{g}}(V \otimes M, M). \tag{15.2}$$

Now let us recall Proposition 6.9, which says that the action of (derived) twisting functors on (the bounded derived category of) $\mathcal{O}_\lambda$ gives a naïve categorification of the left regular $\mathbb{Z}[\mathbb{S}_n]$-module. Twisting functors commute with projective functors $V \otimes {}_-$, see Proposition 6.4. Further, twisting functors are Koszul dual of shuffling functors, see Theorem 10.13. Since projective functors combinatorially preserve right cell modules, shuffling functors, being linear combinations of projective functors, do the same. Taking the Koszul dual switches left and right, which implies that twisting functors combinatorial preserve left cells and hence give a naïve categorification of left cell modules. These are again irreducible as we work with $\mathbb{S}_n$. Applying twisting functors (or, more accurately, the subfunctors Q_s from Proposition 6.8) to (15.2), one obtains the following:

Proposition 15.7 ([MS4])**.** *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be dominant and regular. Let $x, y \in \mathbb{S}_n$ be in the same left cell. Then*

$$\mathcal{L}(L(x \cdot \lambda), L(x \cdot \lambda)) \cong \mathcal{L}(L(y \cdot \lambda), L(y \cdot \lambda)).$$

Combining Proposition 15.7 with Theorem 15.6 we obtain:

Theorem 15.8 ([MS4])**.** *Let $\lambda \in \mathfrak{h}^*_{\mathrm{dom}}$ be dominant and regular and $x \in \mathbb{S}_n$. The answer to Kostant's problem for $L(x \cdot \lambda)$ is an invariant of a left cell.*

Theorem 15.8 does not generalize to other types. Thus in type B_2 with simple reflections s and t we have $s \sim_L st$, however, Kostant's problem has positive answer for $L(s \cdot \lambda)$, see [Ma1], and negative answer for $L(st \cdot \lambda)$, see [Jo3].

15.3. **Structure of induced modules.** We would like to finish with the following classical question for which the idea of categorification gives a new interesting insight. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{p}$, and $\mathfrak{a}$ be the Levi factor of $\mathfrak{p}$. Then any (simple) $\mathfrak{a}$-module V can be trivially extended to a $\mathfrak{p}$-module via $\mathfrak{n}V := 0$. The induced module

$$M(\mathfrak{p}, V) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p})} V$$

is called the *generalized Verma module* associated to $\mathfrak{p}$ and V. Taking $\mathfrak{p} = \mathfrak{b}$ gives usual Verma modules. Taking V finite dimensional, produces parabolic Verma modules. In fact, starting from a module in the category $\mathcal{O}$ (for the algebra $\mathfrak{a}$) produces a module in the category $\mathcal{O}$ (now, for the algebra $\mathfrak{g}$). The structure of usual Verma modules is combinatorially given by Corollary 7.13 (that is, by Kazhdan-Lusztig polynomials). A natural problem is:

Problem 15.9. Describe the structure of $M(\mathfrak{p}, V)$ for arbitrary V.

The general case of this problem is still open, however, some important special cases are settled. The first difficulty of the problem lies in the fact that simple modules over simple Lie algebras are not classified (apart from the algebra $\mathfrak{sl}_2$, which was done by R. Block, see [Ma7, Section 6]). So, at the first stage several authors studied special cases of Problem 15.9 for some known classes of simple modules V, see e.g. [MiSo, FM, MO1] and references therein (see also the paper [MS4] for an overview of the problem).

The first really general result came in [KhMa1], where it was shown that in the case when the semi-simple part of $\mathfrak{a}$ is isomorphic to $\mathfrak{sl}_2$, an essential part of the structure of $M(\mathfrak{p},V)$, called the *rough structure*, does not really depend on the module V but rather on its annihilator, that is a primitive ideal of $U(\mathfrak{g})$. In [MS4] this idea was further developed to obtain the strongest, so far, result on Problem 15.9. Here is a very rough description how it goes:

Instead of looking at $M(\mathfrak{p},V)$ we consider this module as an object of a certain category $\mathcal{A}$, which is a kind of "smallest reasonable category" containing $M(\mathfrak{p},V)$ and closed under projective functors. The category $\mathcal{A}$ has the form $\mathcal{O}(\mathfrak{p},\mathcal{C})$ for some $\mathfrak{p}$ and $\mathcal{C}$ as in Subsection 9.4. One shows that the category $\mathcal{A}$ decomposes into blocks and each block is equivalent to a module category over a standardly stratified algebra. The original module $M(\mathfrak{p},V)$ is related to a proper standard object with respect to this structure. Unfortunately, this relation is not an equality. In general, the corresponding proper standard object only surjects onto $M(\mathfrak{p},V)$. But the kernel, which may be nonzero, is always "smaller" in the sense that it has a filtration by generalized Verma modules induced from simple modules having *strictly bigger annihilators* than V.

Now, the main observation is that, using the action of projective functors on $\mathcal{A}$, one can prove the following:

Theorem 15.10 ([MS4])**.** *Let $\mathfrak{g}=\mathfrak{gl}_n$.*

(*a*) *The action of projective functors on a (regular) block of $\mathcal{A}$ categorifies an induced cell module, where the cell in question is determined (up to a choice inside the corresponding two-sided cell) by the annihilator of V.*
(*b*) *Every (regular) block of $\mathcal{A}$ is equivalent to a subcategory of $\mathcal{O}$ appearing in Theorem 9.13, where induced cell modules were categorified.*

Since $M(\mathfrak{p},V)$ has categorical description as a proper standard module, any equivalence from the above reduces (a part of) the question about the structure of $M(\mathfrak{p},V)$ to the corresponding question for the image of $M(\mathfrak{p},V)$ in $\mathcal{O}$. The latter can be solved using Kazhdan-Lusztig's combinatorics (similarly to how it is done for usual Verma modules). However, the categorical picture completely disregards simple subquotients of $M(\mathfrak{p},V)$ which are induced from modules with strictly bigger annihilators, so this part of the structure of $M(\mathfrak{p},V)$ will be lost. What we really can see using the above picture is simple subquotients of $M(\mathfrak{p},V)$ induced from modules with "comparable" annihilators. This is what is called the *rough structure* of $M(\mathfrak{p},V)$. However, in many cases, for example, in those studied in [MiSo, MO1], it is known that the rough structure coincides with the full structure of the module.

16. Exercises

Exercise 16.1. (*a*) *Prove Lemma 1.2.*

(*b*) *Construct an example of an algebra A such that the groups $\varphi([\mathcal{P}(A)])$ and* [A-mod] *(notation from Subsection 1.2) have different ranks.*
(*c*) *Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. Show that* $[\mathscr{A}/\mathscr{C}]\cong[\mathscr{A}]/[\mathscr{C}]$.
(*d*) *Consider the algebra $U(\mathfrak{sl}_2)$ with the generating system* $\mathbf{e}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\mathbf{f}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$. *Let $(B\text{-mod},\varphi,\mathrm{E},\mathrm{F})$ be a naïve categorification of a simple finite dimensional A-module. Show that for every simple B-module L the element $\varphi^{-1}([L])$ is a weight vector.*
(*e*) *Describe all weak categorifications of the algebra $\mathbb{C}[a]/(a^2-a)$ with involution $a^*=a$.*
(*f*) *Construct a categorification $\mathscr{C}$ for the semigroup algebra of a finite semigroup S with involution $*$ such that $*$ is categorified by an anti-autoequivalence $\circledast$ of $\mathscr{C}$*

and in every 2-representation of $\mathscr{C}$ the anti-autoequivalence $\circledast$ would correspond to taking the biadjoint functor.

(g) *Prove Proposition 3.5.*

(h) *Let $\mathscr{C}$ be a fiat category and $\mathcal{R}_1$ and $\mathcal{R}_2$ be two right cells of $\mathscr{C}$ which do not belong to the same two-sided cell. Show that the cell modules $\mathbf{C}_{\mathcal{R}_1}$ and $\mathbf{C}_{\mathcal{R}_2}$ are not isomorphic.*

(i) *Construct explicitly a "minimal" fiat-category $\mathscr{C}$ categorifying the algebra $\mathbb{C}[a]/(a^2-a)$ with involution $a^*=a$. Construct also cell modules for $\mathscr{C}$.*

Exercise 16.2. (a) *Prove that $M(\lambda)$ is the maximal quotient of $P(\lambda)$ having the property Theorem 4.2*(b).

(b) *Prove Corollary 4.6.*

(c) *Let A be a quasi-hereditary algebra. Prove that both the images of indecomposable projective modules and the images of indecomposable tilting modules in* $[A\text{-mod}]$ *form a basis there.*

(d) *Prove that projective functors commute with $\star$.*

(e) *Let $w\in W$ and s be a simple reflection. Show that $ws<w$ implies that $\theta_s\circ\theta_w\cong\theta_w\oplus\theta_w$. Show further that $ws>w$ implies that $\theta_s\circ\theta_w$ has a unique direct summand isomorphic to θ_{ws}.*

(f) *For integral, regular and dominant λ show that $\mathrm{C}_{w_o}P_\lambda\cong T_\lambda$ and deduce Ringel self-duality of $\mathcal{O}_\lambda$.*

(g) *Prove that $\mathrm{Z}_s\cong\star\circ\hat{\mathrm{Z}}_s\circ\star$, where $s\in W$ is a simple reflection.*

(h) *In the case $\mathfrak{g}=\mathfrak{gl}_2$ show that $\mathrm{T}_s\neq\mathrm{C}_s$ despite of the fact that $\mathrm{T}_sM=\mathrm{C}_sM$ for any $M\in\mathcal{O}_0$.*

(i) *In the notation of Subsection 6.4 show that $\mathrm{R}_s^2\cong\mathrm{P}_s$.*

Exercise 16.3. (a) *Prove Theorem 7.2 using* (7.1).

(b) *Deduce Corollary 7.4 from the double centralizer property and Theorem 7.3.*

(c) *Prove Lemma 7.9.*

(d) *For $n=3$ compute the complete multiplication table of $\mathbb{H}$ in the Kazhdan-Lusztig basis.*

(e) *Check that two different categorifications of the two-dimensional Specht $\mathbb{S}_3$-module (using different parabolic subalgebras of $\mathfrak{gl}_3$) are equivalent.*

(f) *Compute explicitly the algebra categorifying the two-dimensional simple $\mathbb{S}_3$-module.*

(g) *In the notation of Subsection 9.1 let $\mathcal{R}$ be a right cell and s a simple reflection such that $sw<w$ for any $w\in\mathcal{R}$. Show that the functor Q_s is exact on $\mathcal{O}_\lambda^{\hat{\mathcal{R}}}/\mathcal{C}$.*

(h) *Prove Lemma 9.9.*

(i) *Check that both $\mathcal{K}$ and $\mathcal{K}'$, as defined in Subsection 9.4, are closed under the action of projective functors.*

Exercise 16.4. (a) *Show that there are no nontrivial homotopies between linear complexes of projective modules.*

(b) *Determine explicitly all indecomposable objects in the category $\mathcal{LC}(B_0)$ in the case $n=2$ and verify Koszul self-duality of B_0 in this case.*

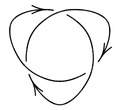
(c) *Verify Theorem 10.13 in the case $n=2$ using an explicit calculation. In particular, determine explicitly the "appropriate shift in grading and position".*

(d) *Prove that $[-a]=-[a]$; $[a]\in\mathbb{Z}[v,v^{-1}]$; $\begin{bmatrix}a\\n\end{bmatrix}=(-1)^n\begin{bmatrix}-a+n-1\\n\end{bmatrix}$; $\begin{bmatrix}a\\n\end{bmatrix}=\frac{[a]!}{[n]![a-n]!}$; $\begin{bmatrix}a\\n\end{bmatrix}\in\mathbb{Z}[v,v^{-1}]$.*

(e) *Compute explicitly the categorifications of the modules $\mathcal{V}_1^3$ and $\mathcal{V}_3$.*

(f) *Show that Serre subcategories defined via increasing Gelfand-Kirillov dimension define a filtration of $\mathcal{V}_1^{\otimes n}$ whose subquotients are isotypic components of $\mathcal{V}_1^{\otimes n}$.*

(*g*) *Compute Jones polynomial for the* trefoil knot

$T :=$ 

(*h*) *Prove Proposition 12.3.*

(*i*) *Check that the endomorphism of* $\hat{\mathcal{V}}_1^2$*, associated to the right crossing, equals* $-v\mathrm{Id} + \cup \circ \cap$*, where* $\cup$ *and* $\cap$ *denote morphisms associate to the cup and cap diagrams, respectively. Find a similar expression for the morphism associated to the left crossing.*

(*j*) *Show that the ranks of* $[\mathcal{O}_\lambda^{\max}]$ *and* $[\mathcal{O}_0^{\max}]$ *from Proposition 12.9 coincide.*

(*k*) *Check that in the case* $n = 2$ *the combinatorics of translations to and from the walls and that of shuffling and coshuffling functors corresponds to the combinatorics of morphisms between tensor powers of* $\mathcal{V}_1$ *as described in Subsection 12.3.*

Exercise 16.5. (*a*) *Classify all* $\mathfrak{sl}_2$*-categorifications of the simple* 1*- and* 2*-dimensional* $\mathfrak{sl}_2$*-modules.*

(*b*) *Prove Lemma 13.15.*

(*c*) *Given an* $\mathfrak{sl}_2$*-categorification, show that there is a filtration of* $\mathcal{A}$ *by Serre subcategories such that subquotients of this filtration descend exactly to isotypic* $\mathfrak{sl}_2$*-components in* $[\mathcal{A}]$*.*

(*d*) *Prove Proposition 14.2 in the case* $\mathbb{F} = \mathbb{C}$*.*

(*e*) *Let* M *be an* $\mathbb{F}[\mathbb{S}_n]$*-module. Show that*

$$\mathrm{ch}\, M = \sum_{\mathbf{i}\in\mathbb{F}_p^n} a_{\mathbf{i}} \exp(\mathbf{i}) \quad \textit{implies} \quad \mathrm{ch}\,(\mathrm{E}_i\, M) = \sum_{\mathbf{i}\in\mathbb{F}_p^{n-1}} a_{(\mathbf{i},i)} \exp(\mathbf{i}).$$

(*f*) *Compute explicitly all blocks and the corresponding defect groups for* $\mathbb{S}_3$ *over an algebraically closed field of characteristic* 2 *and* 3*.*

(*g*) *Prove Theorem 14.14*(a) *on the level of the Grothendieck group.*

(*h*) *Show that* $\{f_w : w \in \mathbb{S}_n\}$ *is not a basis of* $\mathbb{Z}\mathbb{S}_n$ *for* $n > 1$*.*

(*i*) *Prove Proposition 15.7 in the case* $n = 3$ *by a direct computation.*

(*j*) *Prove that the generalized Verma module induced from a Verma module is again a Verma module.*

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