AN INTRODUCTION TO $\mathfrak{sl}_n$-LINK HOMOLOGIES AND CATEGORIFICATION

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Abstract. The Jones polynomial is a celebrated invariant of links. It was introduced by Vaughan Jones around 1985 and it is more than just a strong invariant of links. That is, it appears naturally in former unrelated fields of mathematics and theoretical physics. Jones has opened a new field of research sometimes called quantum topology.

And he started the so-called "Jones revolution": Shortly after Jones others discovered a bunch of new link polynomials by "reformulating" Jones ideas. Before Jones there was a lack of link polynomials and after Jones there where "too many". The main question after Jones was how to "order" them.

One way to "explain" its appearance is the usage of the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$. Hence the name $\mathfrak{sl}_2$-link polynomial.

In 1999 Khovanov categorified the Jones polynomial. He gave a chain complex whose homology decategorifies to the Jones polynomial. History repeats itself: Khovanov's construction appear naturally in different fields and he opened again a new field of research that is a huge business nowadays. In fact, history really repeats itself: Before Khovanov there was a lack of link homologies and after Khovanov there where "too many".

Since the Jones polynomial can be "explained" using the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$ one should expect that a categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ would "explain" Khovanov homology.

That is one reason why Khovanov and Lauda (and independently Rouquier) introduced $\mathcal{U}(\mathfrak{sl}_n)$ that categorifies $\mathbf{U}_q(\mathfrak{sl}_2)$ (and open yet another field of research).

The approach of this lecture is to explain the construction of the Khovanov homology in detail in the first part and show how tit connects to "higher" representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$ in the second part of the lecture.

We will completely focus on the $\mathfrak{sl}_2$ case but everything works in more generality.

Outline.

- 1a We start on the uncategorified level and explain Kauffman's approach to calculate the Jones polynomial.
- 2a We explain how Khovanov has categorified the Jones polynomial using his original, algebraic definition.
- 3a We give re-interpretation of Khovanov's approach using 1+1 dimensional TQFT's (do not be scared: These are mostly pictures).
- 1b We start on the uncategorified level again and explain how the Jones polynomial can be "explained" using symmetries of $\mathbf{U}_q(\mathfrak{sl}_2)$.
- 2b We explain how Khovanov and Lauda have categorified $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ using a diagrammatic category $\mathcal{U}(\mathfrak{sl}_n)$.
- 3b We illustrate how Khovanov homology is an instance of "higher" $\mathcal{U}(\mathfrak{sl}_n)$-representation theory.

A remark. Khovanov homology and categorification of quantum groups is a modern and very active field of research which is developing very fast at the moment.

Depending on the interest of the participants: I can explain the far-reaching connections of $\mathcal{U}(\mathfrak{sl}_n)$ in a follow-up of this lecture.

Requirements. The intended audience is PhD students with some minimal background in low-dimensional topology (especially knot theory), algebraic topology (some basic homological algebra), category theory (basic understanding of categories, functors, natural transformations etc.) and representation theory (representation of algebras).

All is not mandatory in the sense that, depending on the background of the audience, I can explain some of it and give references for further reading.

References

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