English version

Higgs Bundles and the Geometric Langlands Program

Level of course
PhD Course

Semester/quarter
3rd + 4th quarter (Spring 2010)

Hours per week
4

Name of lecturer
Jørgen Ellegaard Andersen, Peter Gothen, Johan Marten and Gor Sarkissian

Prerequisites
Please contact the teacher

Course contents

Introduction:

The Langlands program started in the late '60s when Robert Langlands conjectured deep correspondences between certain number-theoretic statements, involving representations of the Galois group of a number field, and p-adic harmonic analysis, involving automorphic representations of the adeles of the field. Some particular instances of these Langlands conjectures have been verified (such as the Shimura-Taniyama-Weil conjecture that implies Fermat's last theorem), but in general they are still wide open. Much more progress has been made in a variation on the Langlands conjectures where one replaces the number fields occurring by function fields of algebraic curves over finite fields. Drinfeld and Lafforgue both received their Fields medals for work in this function field context.

Starting in the 80s through work of Drinfeld and Beilinson, the results involving functions fields of curves over finite fields led by analogy to another set of conjectures, involving complex algebraic geometry. This so-called geometric Langlands program has been the focus of intense activity in the last decade or so. In addition, in recent years Witten and collaborators have shown how the geometric Langlands program is related to the so-called S-duality, a conjectural property of quantum Yang-Mills theories which can be understood as a generalization of electric-magnetic duality.

An enormous amount of current mathematical research is related to various aspects of these programs, but the technical background needed for understanding even the statements of the geometric Langlands program can be quite daunting. The modest goal of this course is to work through the paper of Kapustin and Witten, developing the necessary background as we go along. It is our hope that this course would give participants enough insight in both the geometric program and its connection with physics so as to serve as a source for future research projects.

References: Kapustin, Anton; Witten, Edward, Electric-magnetic duality and the geometric Langlands program. Commun. Number Theory Phys. 1 (2007), no. 1, 1–236. Ben-Zvi, David, Lectures on the Geometric Langlands Correspondence. London Math Society Lecture Note Series, Cambridge University Press, to appear. Frenkel, Edward, Recent advances in the Langlands program. Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 2, 151–184 (electronic). Frenkel, Edward, Gauge Theory and Langlands Duality, Seminaire Bourbaki talk.

Spinors:

We shall begin gently by discussing spinors. Already showing some early appearances in the 19th century, spinors were first invented by Cartan in 1913. They were rediscovered by Dirac, who sought an operator that squared to the Laplacian for his work on a quantum theory of the electron. The Dirac operator also arose in the work of Atiyah and Singer on the index theorem, who placed spinors in the differential geometric context still used today. We will begin by introducing Clifford algebras and use them to define the Spin and Spin^c groups and describe their representations and relations to the special orthogonal groups. We'll then switch to geometry and discuss obstructions to Spin-structures on pseudo-Riemannian manifolds, spinor bundles associated with Spin structures, the Dirac operator and the LichnerowiczÐWeitzenbck formula. Time permitting we'll say a few things about the index theorem and Bott periodicity as well.

References: Hitchin, Nigel, The Dirac operator. Invitations to geometry and

topology, 208–232, Oxf. Grad. Texts Math., 7, Oxford Univ. Press, Oxford, 2002. Lawson, H. Blaine, Jr.; Michelsohn, Marie-Louise Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

Physics Background:

The course will then turn towards reviewing the evidences for conjecturing that Langland duality can be realized as S duality in quantum field theory. At the beginning we describe Magnetic monopoles in non abelian Yang-Mills theory and compute magnetic weights. We show that while electric charge takes values in the weight lattice of the gauge group, magntic charge take values in the weight lattice of the Langland dual group. This provides us with the first evidence that Langland duality should be kind of duality interchanging electric and magnetic charges. Such a duality was proposed and called S-duality in the context of the N=4 D=4 Super Yang-Mills theory. This brings us to the next topic of the course: N=4 Super Yang-Mills theory. At the beginning we introduce all the ingredients necessary for construction of the N=4 D=4 SYM: Superalgebras, Grassmanian variables and Supersymmetry. Then we will construct N=4 D=4 SYM as dimensional reduction of N=1 D=10 SYM, and analyze the S-duality. The next topic of the course is Topological field theory. We will give general introduction to the topic and explain the tool of topological twisting converting supersymmetric theories to topological theories. After that we will apply topological twisting to N=4 D=4 SYM and derive one-parametric family of topological theories. Supersymmetry transformations and Topological Lagrangian will be derived.

Considering compactification of the derived at the previous point Topological field theory to two dimension we will derive the main object of the course: two-dimensional Sigma Model with target space given by the Hitchin's Moduli Space.

References: P.Goddard, J. Nuyts and D.I.Olive, "Gauge Theories And Magnetic Charge," Nucl. Phys. B 125 (1977) 1. M.R.Douglas, "D-branes, categories and N = 1 supersymmetry," J. Math. Phys. 42 (2001) 2818 E.Witten, "Introduction To Cohomological Field Theories," Int. J. Mod. Phys. A 6 (1991) 2775.

Higgs bundles and the Hitchin system:

Higgs bundles were introduced in 1987 by Hitchin, who was interested in what happened to the self-duality equations for connections on R^4 when one imposes translation-invariance in two dimensions. The remaining conditions on R^2, known as the Hitchin equations, are conformally invariant and can therefore be studied on any Riemann surface, where they are interpreted as equations for the curvature on a vector bundle on the surface, together with a 1-form valued endomorphism of the bundle, the so-called Higgs field. Algebraically they can be understood as holomorphic vector bundles with a holomorphic Higgs field, to which a natural stability notion applies. One of the remarkable facts about their moduli-spaces is that they are hyper-Kaehler and very naturally come equipped with an algebraically completely integrable system. Furthermore, through work of Corlette, Donaldson, Hitchin and Simpson on harmonic metrics the other complex structures on these moduli spaces can be interpreted as arising from different moduli-problems: reductive representations of the fundamental group of the surface, or flat reductive connections. This correspondence also largely extends to higher dimensions and through the work of Simpson is furthermore linked with variations of Hodge structure. We shall introduce all of these concepts. In the last two decades, through the work of Beilinson, Drinfeld and others moduli spaces of Higgs bundles have been found to play a central role in the geometric Langlands program. The Hitchin fibration (in positive characteristic) also played a key role in the recent proof by Ngo of the fundamental lemma, a crucial component of the number-theoretic Langlands program that had defied any solution for 25 years.

References: Hitchin, N. J. The self-duality equations on a Riemann surface. Proc. London Math. Soc. (3) 55 (1987), no. 1, 59–126. Hitchin, Nigel Stable bundles and integrable systems. Duke Math. J. 54 (1987), no. 1, 91–114. Appendix to: Wells, Raymond O., Jr. Differential analysis on complex manifolds. Third edition. With a new appendix by Oscar Garcia-Prada. Graduate Texts in Mathematics, 65. Springer, New York, 2008. xiv+299 pp. Simpson, Carlos T. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Amer. Math. Soc. 1 (1988), no. 4, 867–918.

Witten and Kapustin's interpretation of Langlands Duality:

-The remaining part of the course will be devoted to the analyze of the action of the S-duality of original 4D theory in the two-dimensional Sigma model.

To analyze this action we consider Sigma model on a world-sheet with boundary and introduce the next two notions: branes and loop operators. Branes roughly can be described as submanifold with some additional sheaf structure where boundary of Sigma model is mapped to. Loop operator is an operation which upon acting on boundary conditions produces new boundary conditions, and thus transforming branes to branes. Using Hitchin fibration we will show that reduction of S-duality to two dimension will act as T-duality and transform 0-brane, given by skyscraper sheaf, to a brane supported on a fiber of the Hitchin's Moduli Space of Langland dual group, thus establishing that Hitchin's moduli spaces for dual groups are Hitchin fibrations with dual fibers. At the next step we study Wilson lines and t'Hooft opertors of original D=4 SYM, and show that upon reduction to two dimensions they produce described above loop operators. We also will show that Wislon line and t'Hooft operator are S-dual to each other. We will show that 0-brane is eigensheaf (called electric eigenbrane) of the reduction of the Wilson line, and fiber supported brane is eigensheaf (called magnetic eigenbranes) of the reduction of the t'Hooft opeartor. To complete the Langland duality we will show that reduction of t'Hooft operators correspond to Hecke operators and magnetic eigenbranes correspond to D-modules.

N.J.Hitchin, A.Karlhede, U.Lindstrom and M.Rocek, "Hyperkahler Metrics and Supersymmetry," Commun. Math. Phys. 108 (1987) 535. A.Strominger, S.T.Yau and E.Zaslow, "Mirror symmetry is T-duality," Nucl. Phys. B 479 (1996) 243

Teaching methods
4 hours of lectures per week

Assessment methods
Passed / not passed will be based on the students participation in the course.

Credits
10 ECTS

Language of instruction
Danish or/and English

Course enrolment
Please send an e-mail to Maiken Nielsen, maiken@imf.au.dk