PHD Course - FALL 2014
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CLUSTER VARIETIES AND GHK MIRROR SYMMETRY

Synopsis. Cluster algebras were originally defined combinatorially Fomin and Zelevinsky, with the hope that they would eventually lead to a better conceptual understanding of Lusztig's dual canonical bases and positivity. They have been gaining an increasingly significant role in several areas of math and physics. Fock and Goncharov interpreted these algebras geometrically, and they defined *cluster varieties*, which are constructed by gluing algebraic tori together via certain birational maps, called mutations. They conjectured the existence of certain dual canonical bases of global functions on these cluster varieties. These conjectures were recently proven (with slight modifications) by Gross, Hacking, Keel, and Kontsevich (GHKK), using ideas from mirror symmetry. The construction of these canonical "theta functions" is expected to have deep significance in several areas of math, including representation theory, symplectic geometry, and Teichmüller theory.

Goals. The main goal of this coures is to understand the GHKK construction mentioned above. Along the way, we will cover:

- The basics of cluster algebras and cluster varieties, including their constructions and some examples relevant to representation theory and Teichmüller theory.
- The birational geometric description of cluster varieties due to Gross, Hacking, and Keel.
- Tropicalizations of cluster varieties.
- Scattering diagrams and broken lines (combinatorial objects which morally record data about holomorphic disks in the cluster varieties).
- GHKK's construction and their proof of the Positive Laurent Phenomenon.
- Motivation from mirror symmetry, including the homological interpretation of theta functions and the SYZ picture of dual special Lagrangian torus fibrations.

Time permitting, we may investigate some related topics, such as tropicalizations of theta functions, mirror symmetry conjectures for log Calabi-Yau varieties, other aspects of the Gross-Siebert mirror symmetry program, and canonical bases in representation theory.

Requirements. The GHKK construction is quite elementary, and very little background is needed. Knowing some basic algebraic geometry will be helpful (concepts like how certain rings determine varieties and what blowing up is), but these things can be briefly explained as needed. In particular, we will begin with an introduction to toric varieties, which will then be used throughout the course.

Some Useful References.

- V. Fock and A. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci.Éc. Norm. Sup. (4) **42** (2009), no. 6, 865-930, also at arXiv:math/0311245v7.
- W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.
- M. Gross, *Tropical Geometry and Mirror Symmetry*, CBMS Regional Conference Series in Mathematics, vol. 114, A.M.S., 2011.
- M. Gross, P. Hacking, S. Keel, *Birational Geometry of Cluster Algebras*, arXiv:1309.2573.
- M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, arXiv:1106.4977.
- M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Mirror symmetry and cluster varieties*, (in preparation).