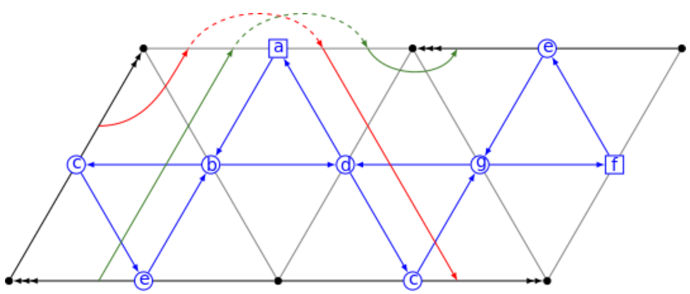
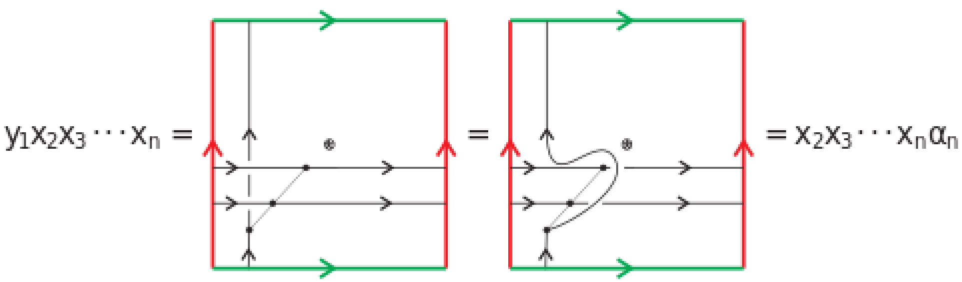
Unified quantization of character varieties
David Jordan, University of Edinburgh

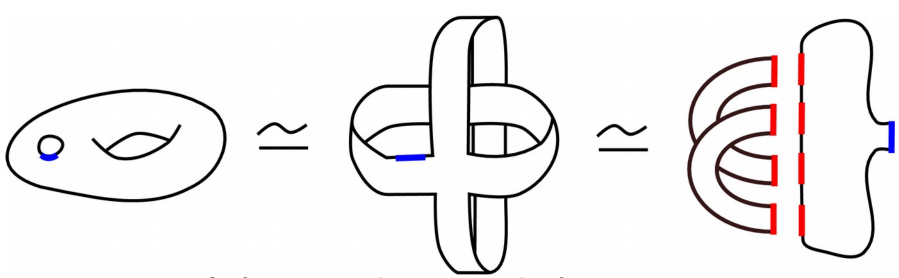
$$Z(S) = \int_S \mathrm{Rep}_q(G)$$



Fock-Goncharov
quantum cluster algebras



Skein algebras



Alekseev-Grosse-Schomerus
moduli algebras

Classical character vareities

- G = reductive algebraic group, e.g. $G = SL_N$
- S = surface, M = 3-manifold.
- The G-character variety of S is:

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- Atiyah-Bott-Goldman Poisson bracket
  - Tangent space: $T_E(Ch(S)) \cong \Omega^1(S, \mathfrak{g})$
  - Poincare: $\Omega^1(S, \mathfrak{g}) \otimes \Omega^1(S, \mathfrak{g}) \to \Omega^2(S, \mathfrak{g} \otimes \mathfrak{g}) \to \mathfrak{g} \otimes \mathfrak{g}$
  - Killing form: $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$

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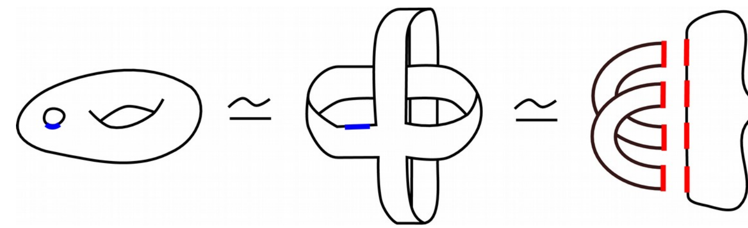
  - Killing form: $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$
- Universal property/compatibility with pullback
- Lagrangians from 3-manifolds with boundary

Outline

- Recall three well-known quantizations: Alexeev-Grosse-Schomerus, Fock-Goncharov, and skein modules.
- Define universal quantizations.
- Recover well-known schemes from the univeral one.
- Construct extended 3D&4D topological field theories.
- (Expected) relations to WRT/Hennings theory.
- Special phenomena at roots of unity.

Quantizations of character varieties



- Moduli algebra quantizations
  - Fock-Rosly: compute Poisson bracket using ribbon graph presentation of S and classical r-matrices
  - AGS: Quantize Fock-Rosly bracket using quantum R-matrices

$$\sum_{j,m} l^i_j R^{jk}_{lm} \partial^m_n = \sum_{o,p,r,t,u,v} R^{ik}_{op} \partial^p_r R^{ro}_{tu} l^u_v R^{vt}_{ln},$$

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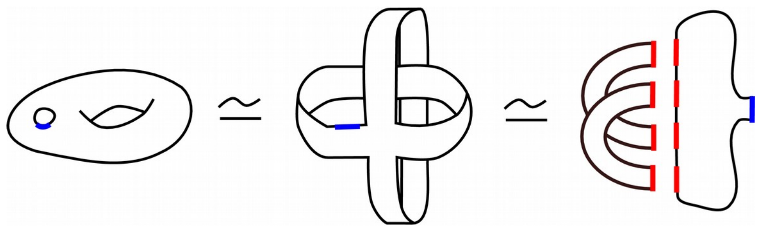

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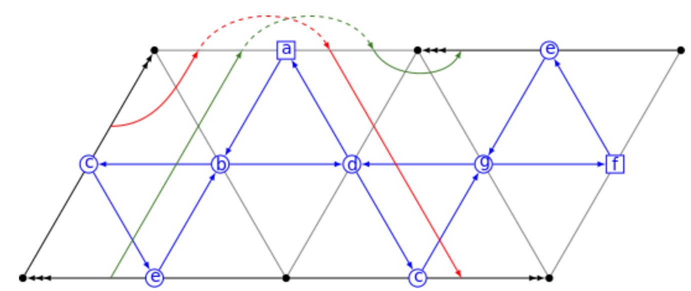
- Quantum cluster algebras
  - Fock-Goncharov: Instead consider framed character varieties, with reduction to Borel subgroups B along boundaries.
  - Triangulation → Cluster variables $\{x_i, x_j\} = a_{ij} x_i x_j$
  - Quantization → $x_i x_j = q^{a_{ij}} x_j x_i$



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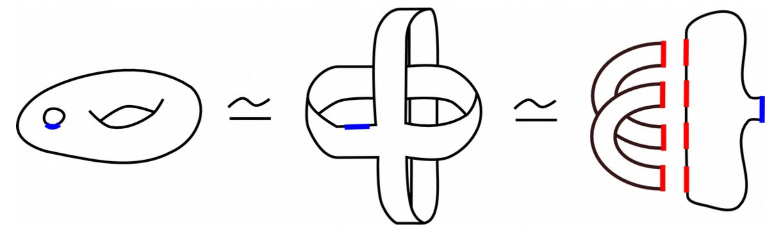

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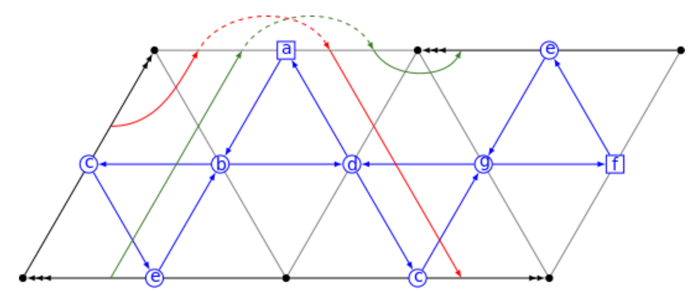

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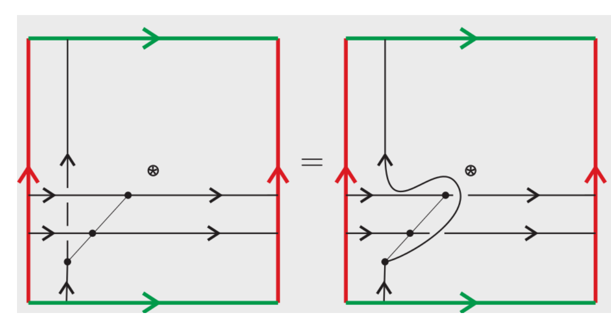

- Skein algebra
  - Sk(M) = Vector space spanned by all tangles in M, modulo local "skein" relations from $\mathrm{Rep}_q(SL_2)$
  - Sk(S) = Sk(S x I) = Algebra under concatenation/ superposition operation.

Choices, choices, choices...

- Each quantization scheme above involved making choices (ribbon graph, triangulation, skein presentation) in the definition.
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- And yet, classical character varieties are **universal**:
  - Functoriality: $i : S_1 \hookrightarrow S_2 \rightsquigarrow Ch(S_2) \to Ch(S_1)$
  - Excision: $Ch(S_1 \bigcup_{P \times I} S_2) = Ch(S_1) \underset{Ch(P \times I)}{\times} Ch(S_2)$
  - Normalization: $Ch(D^2) = pt/G$

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- It is difficult to compute globally with FG/Skeins.
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  - Functoriality: $i : S_1 \hookrightarrow S_2 \rightsquigarrow Ch(S_2) \to Ch(S_1)$ $\implies Z_{cl}(i) : Z_{cl}(S_1) \to Z_{cl}(S_2)$
  - Excision: $Ch(S_1 \bigcup_{P\times I} S_2) = Ch(S_1) \underset{Ch(P\times I)}{\times} Ch(S_2)$ $\implies Z_{cl}(S_1 \bigcup_{P\times I} S_2) = Z_{cl}(S_1) \underset{Z_{cl}(P\times I)}{\boxtimes} Z_{cl}(S_2)$
  - Normalization: $Ch(D^2) = pt/G$ $\implies Z(D^2) = \mathrm{Rep}(G)$
- Ben-Zvi-Francis-Nadler: $Z_{cl}(X)$ is uniquely determined by these properties.
- Caution: really, this holds for the **character stack**, but we don't distinguish. We have always a global sections functor Γ from the character stack to the character variety.

A universal quantization

- So, we **define a category Z(S)** of "quasi-coherent sheaves" on the "quantum character variety", by requiring:
  - Functoriality: $i : S_1 \hookrightarrow S_2 \rightsquigarrow Z(i) : Z(S_1) \to Z(S_2)$
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- Rigorous construction uses "factorization homology" of Lurie/Ayala-Francis-Tanaka.
- Braid group and mapping class group actions emerge naturally from isotopies of embeddings.
- Z(S) has a distinguished object $\mathcal{O}_S \in Z(S)$ the "quantum structure sheaf" and "global sections functor" Γ.

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- **Theorem** (Ben-Zvi-Brochier-J '15)**:** This recovers and generalizes the AGS algebras.
  - $Z(Ann) = \mathcal{O}_q^{RE}(G) - \mathrm{mod}_{\mathrm{Rep}_q(G)}$ (Hochschild homology, uses Lyubashenko-Majid CoEnd/braided duals)
  - Ribbon graph presentation of S $\rightsquigarrow Z(S) \simeq A_S - \mathrm{mod}_{\mathrm{Rep}_q(G)}$, recovering AGS algebras.

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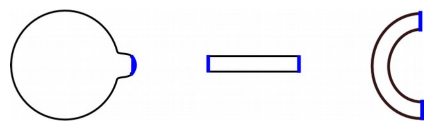
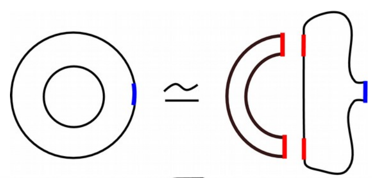
- **Theorem** (BZBJ '16)**:** For S closed surface, Z(S) is the **quantum Hamiltonian reduction** of A_S-mod, for an explicitly given **multiplicative quantum moment map**.
  - Recovers and generalizes Frohman-Gelca: $End_{Z(T^2)}(\mathcal{O}_S) = \mathcal{D}_q(H)^W$
  - Adding "mirabolic"/Ruijenars-Snijder marked point → Type A spherical double affine Hecke algebras. (Balagovic-J '16)

Module structures on Z(S)

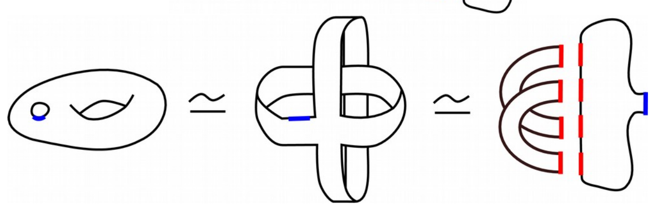
- Choice of disk in boundary of S $\rightsquigarrow \mathrm{Rep}_q(G)$ -module structure on $Z(S)$
- Get an adjoint pair of module functors: $\mathrm{Rep}_q(G) \rightleftarrows Z(S)$
- Choice of boundary **compontent** Z(Ann)-action on Z(S) → quantum moment maps.
- Standard techniques ("Barr-Beck") → compute Z(S) recursively using excision.



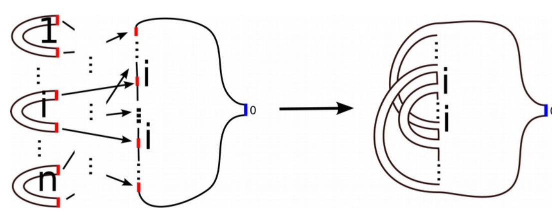
FRT algebra



RE algebra



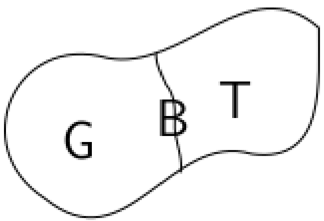
Heisenberg double



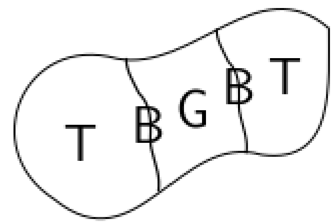
AGS algebras

Quantum Hamiltonian Reduction $\rightsquigarrow \mathcal{D}_q(H)^W$

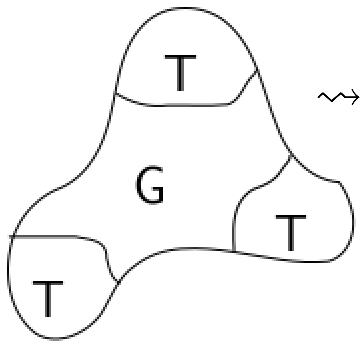
Recovering Fock-Goncharov



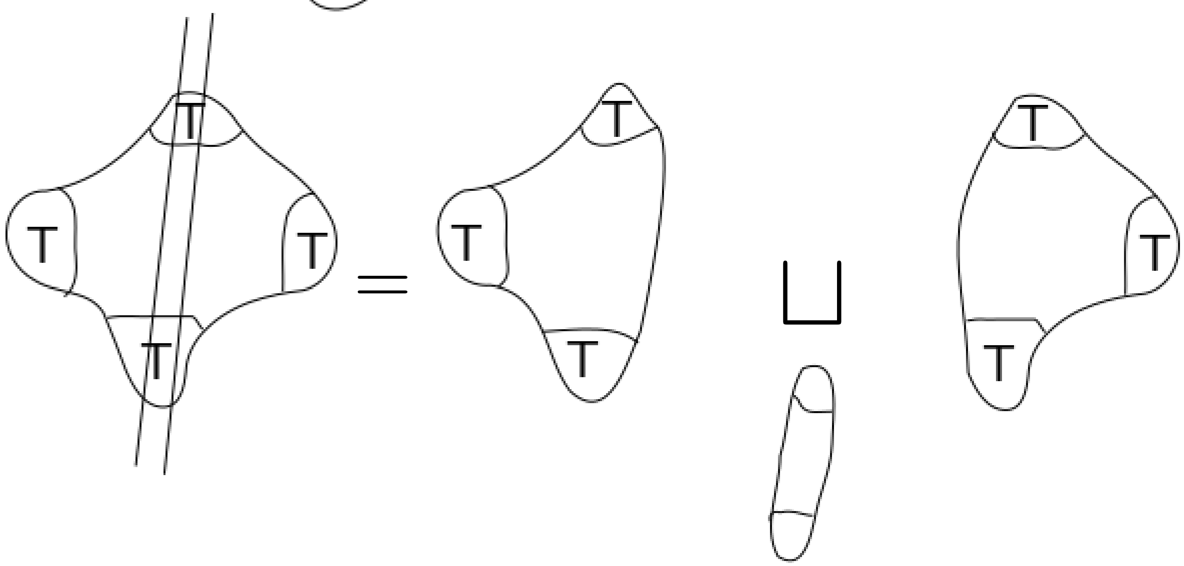
$\rightsquigarrow G\backslash\mathcal{O}_q(G/N)/T$



$\rightsquigarrow G\backslash\mathcal{O}_q(G/N \times G/N)/(T \times T)$
$\supseteq \mathrm{Rep}_q(T)$

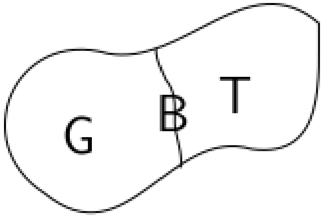


$\rightsquigarrow G\backslash\mathcal{O}_q(G/N \times G/N \times G/N)/(T \times T \times T)$
$\supseteq \mathcal{O}_q(X_{\underline{\omega}}),$ a quantum torus $x_i x_j = q x_j x_i$.



$\rightsquigarrow$ Gluing of quantum torus charts

Recovering Fock-Goncharov



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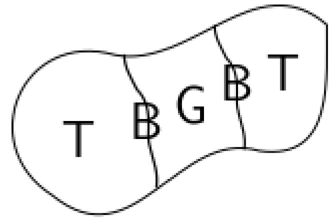

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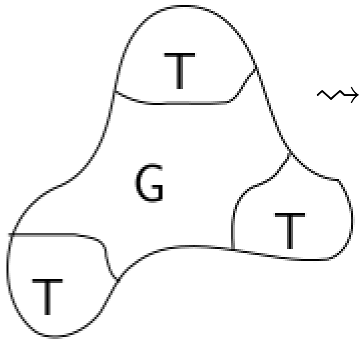

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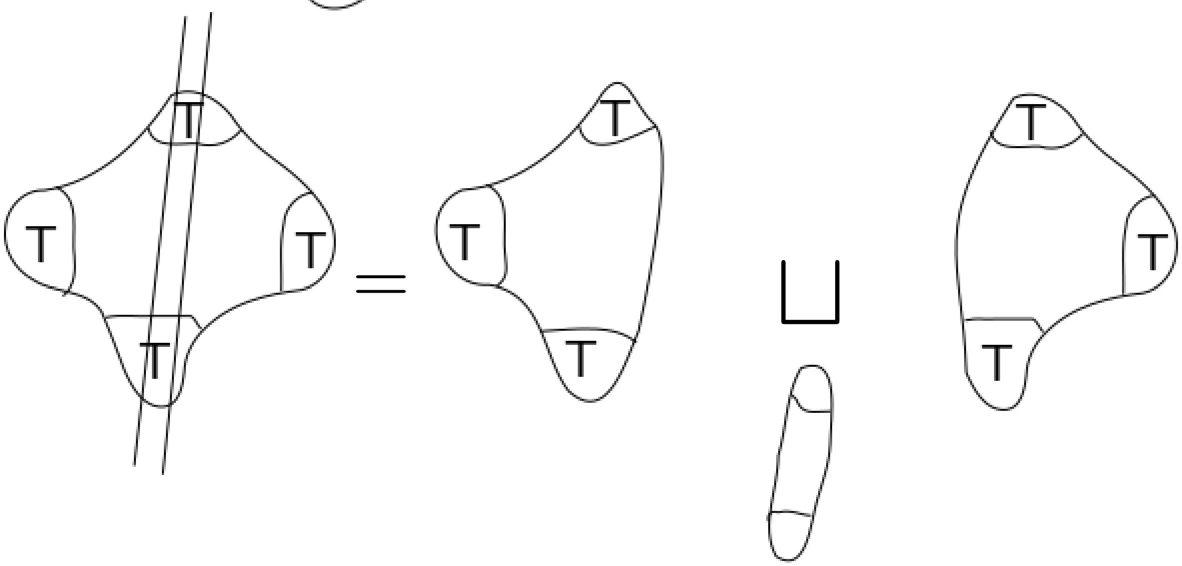


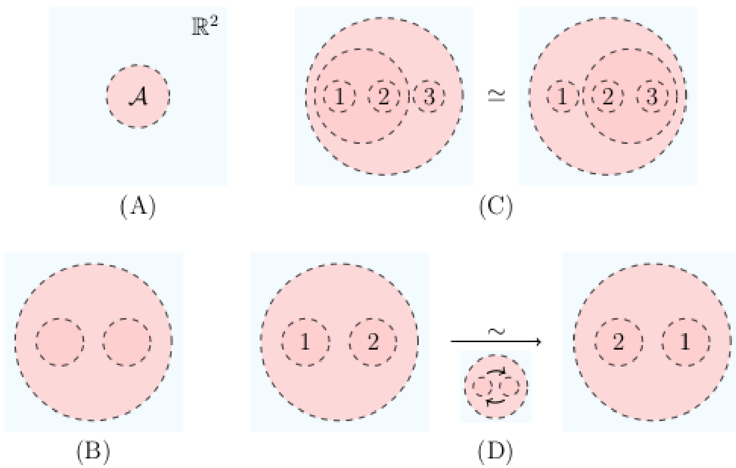
$\rightsquigarrow$ Gluing of quantum torus charts

Theorem (J-Le-Schrader-Shapiro '19): There exist canonical objects $\mathcal{O}_{\underline{\omega},\Delta}$ in stratified quantum character varieties $Z(\widetilde{S})$, and isomorphisms between $End(\mathcal{O}_{\underline{\omega},\Delta})$ and the associated FG chart.

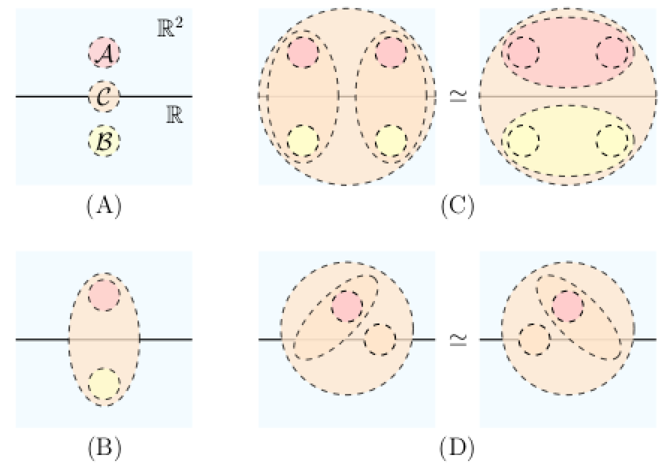
Corollary: AGS and FG quantizations coincide, upon localizing (quantum cluster embeddings of AGS algebras).

Three-dimensional structures

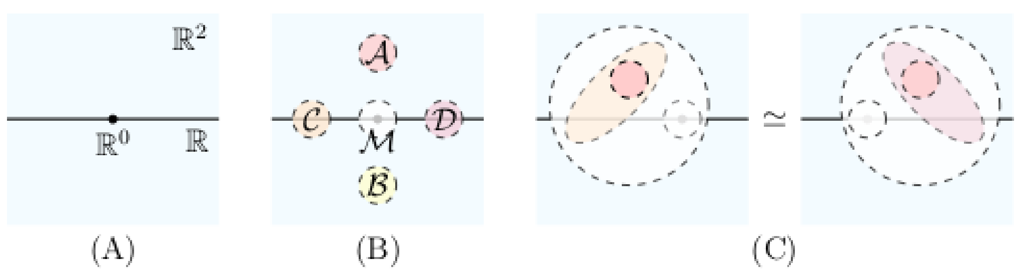
- Haugseng, Johnson-Freyd-Scheimbauer: Braided tensor categories naturally form **BrTens**, a **4-category** (iterated Segal space)



Objects: braided tensor categories



1-morphisms: A-B-central tensor categories



2-morphisms: A-B-central C-D-bimodules

3-morphisms: bimodule functors

4-morphisms: bimodule natural transformations

Three-dimensional structures

- Cobordism hypothesis: fully extended n-dimensional TFT's correspond to n-dualizable objects of BrTens.
- **Theorem** (Brochier-J-Snyder '18)**:** Rigid braided tensor categories with enough compact projectives are 3-dualizable in BrTens.
  - This includes $\mathrm{Rep}_q(G)$ for q generic (semisimple).
  - Also Lusztig's divided powers/restricted $\mathrm{Rep}_q^{Lus}(G)$, for q a root of unity (H.H. Andersen).
  - Also modular tensor category obtained from $\mathrm{Rep}_q^{Lus}(G)$ by quotienting neglibles.
- **Theorem** (BJS '18)**:** Modular tensor categories are 4-dualizable, and invertible (known to Freed-Teleman and Walker in different language).
- Hence by the Corbordism Hypothesis, we obtain a fully local (a.k.a. fully extended) TFT Z: $\mathrm{Bord}^{3+\varepsilon/4} \to$ BrTens. In other words,
  - To closed 4-manifolds W, it assigns numbers Z(W) (in modular case).
  - To closed 3-manifolds M, it assigns a vector space Z(M).
  - To closed surfaces S, it assigns a category Z(S), the quantum character variety.
  - To the circle it assigns a monoidal category Z(S^1) = HH(Rep_q(G))
  - To the point it assigns the braided monoidal category Rep_q(G)

Quantum A-polynomial

- Let K be a knot in S^3, let M denote the knot complement. Since $\partial M = T^2$, M defines functors,

$$Z_+(M) : Z(T^2) \to \mathrm{Vect}, \qquad Z_-(M) : \mathrm{Vect} \to Z(T^2).$$

- Have a global sections functor, $\Gamma = Hom_{Z(T^2)}(\mathcal{O}_S, -) : Z(T^2) \to D_q(H)^W \,\mathrm{mod}$
- In other words, from a knot K, we get a system of difference equations, which q-deforms the classical A-polynomial → canonical construction of (some kind of) quantum A-polynomial.
- Note: Colored Jones J(K) is an element of $Z_{WRT}(T^2)$ obtained in the same way. One can view J(K) as an element of the q-difference system $\Gamma(Z_-(M))$.

Relative field theories, Z, and WRT

- Let **1** denote the trivial TFT in BrTens.
- Definition (Freed-Teleman/Gwilliam-Scheimbauer/Fuchs-Schweigert):
  A **relative field theory** is given by a dualizable morphism $\mathrm{Rep}_q(G)$ to **1** in BrTens.
- Any braided tensor category, regarded as a central algebra over itself defines such a relative field theory.
- Expectation (many people): The WRT 3D TFT is a relative field theory relative to the 4D TFT we constructed above.
- Consequence (of exp.): The colored Jones polynomial J(K) is naturally an element in the "quantum A-polynomial" system we defined above.
- Consequence (of exp.): Alternative construction of Hennings invariant 3D TQFT.

Skein modules and roots of unity

- **Theorem** (Cooke, 18): The subcategory of compact projective objects of Z(S) is the **skein category,** when q is not a root of unity.
  - Corollary: Skein = FG = AGS (quantum trace maps of Bonahon-Wong)
- **Work in progress** (BJS): The skein category is a full (and proper!) subcategory of Z(S), when q is a root of unity. Uses theory of tilting modules.
- **Theorem** (Ganev-J '18): The affine quantum character variety $\Gamma(A_S)$ is an **Azumaya** algebra over the classical character variety. Methods:
  - Use quantum Frobenius functor Rep(G) → Rep_q(G).
  - Use functoriality of quantum character variety.
  - Use Brown-Gordon theory of quantum Poisson orders.
  - Prove that quantum Hamiltonian reduction of Azumaya algebras is Azumaya.
- **Corollary** (combining C, BJS, GJ): Skein algebras at roots of unity are Azumaya algebras over their classical counterparts. (a theorem of Bonahon-Wong for SL_2)

Summary

- Character varieties satisfy a simple universal property with respect to embeddings of surfaces.
- Replacing Rep(G) by $\mathrm{Rep}_q(G)$ in universal property gives universal quantization.
- Making further choices one recovers each of the AGS/FG/Skein module presentation.
- Main tools for computing are excision (topology) and Barr-Beck (rep. theory).
- Gives conceptual explanation for mapping class group symmetry, braid group actions, (certain) cluster transformations.
- Gives extended/fully local 3- or 4-dimensional TFT, possibility to use TFT techniques in studying quantum A-polynomial.
- Subtle interesting behavior at roots of unity – no skein description, but observed Azumaya algebra phenomena.