Unified quantization of character varieties
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\[ Z(S) = \int_S \operatorname{Rep}_q(G) \]

Fock-Goncharov quantum cluster algebras

Skein algebras

Alekseev-Grosse-Schomerus moduli algebras
Classical character varieties

- $G =$ reductive algebraic group, e.g. $G = SL_N$
- $S =$ surface, $M =$ 3-manifold.
- The $G$-character variety of $S$ is:

$$Ch(S) = \{\pi_1(S) \to G\}/G$$

$$= \{G\text{-local systems on } S\}/\text{isom}$$
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- Atiyah-Bott-Goldman Poisson bracket
  - Tangent space: $T_E(Ch(S)) \cong \Omega^1(S, g)$
  - Poincare: $\Omega^1(S, g) \otimes \Omega^1(S, g) \to \Omega^2(S, g \otimes g) \to g \otimes g$
  - Killing form: $g \otimes g \to \mathbb{C}$
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- Universal property/compatibility with pullback
- Lagrangians from 3-manifolds with boundary
Outline

- Recall three well-known quantizations: Alexeev-Grosse-Schomerus, Fock-Goncharov, and skein modules.
- Define universal quantizations.
- Recover well-known schemes from the universal one.
- Construct extended 3D&4D topological field theories.
- (Expected) relations to WRT/Hennings theory.
- Special phenomena at roots of unity.
Quantizations of character varieties

- Moduli algebra quantizations
  - Fock-Rosly: compute Poisson bracket using ribbon graph presentation of $S$ and classical $r$-matrices
  - AGS: Quantize Fock-Rosly bracket using quantum $R$-matrices

\[
\sum_{j,m} l_j R^{jk}_{lm} \partial^m_n = \sum_{o,p,r,t,u,v} R^{ik}_{op} \partial^p_r R^{ro}_{tu} l^u v R^{vt}_{ln},
\]
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• Quantum cluster algebras
  – Fock-Goncharov: Instead consider framed character varieties, with reduction to Borel subgroups B along boundaries.
  – Triangulation → Cluster variables \( \{ x_i, x_j \} = a_{ij} x_i x_j \)
  – Quantization → \( x_i x_j = q^{a_{ij}} x_j x_i \)
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- **Skein algebra**
  - $\text{Sk}(M) = \text{Vector space spanned by all tangles in } M$, modulo local "skein" relations from $\text{Rep}_q(SL_2)$
  - $\text{Sk}(S) = \text{Sk}(S \times I) =$ Algebra under concatenation/superposition operation.
Choices, choices, choices...

- Each quantization scheme above involved making choices (ribbon graph, triangulation, skein presentation) in the definition.

- These choices are obstacles to defining a TFT; it is not enough to show that the construction is independent of choices, it must be functorial.
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- And yet, classical character varieties are universal:
  - Functoriality: \( i : S_1 \hookrightarrow S_2 \leadsto Ch(S_2) \to Ch(S_1) \)
  - Excision: \( Ch(S_1 \bigcup_{p \times I} S_2) = Ch(S_1) \times_{Ch(p \times I)} Ch(S_2) \)
  - Normalization: \( Ch(D^2) = pt/G \)

Caution: really, this holds for the character stack, but we don’t distinguish. We have always a global sections functor \( \Gamma \) from the character stack to the character variety.
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  - Functoriality: \( i : S_1 \hookrightarrow S_2 \leadsto \text{Ch}(S_2) \to \text{Ch}(S_1) \) \( \implies Z_{cl}(i) : Z_{cl}(S_1) \to Z_{cl}(S_2) \)
  - Excision: \( \text{Ch}(S_1 \bigcup_{P \times I} S_2) = \text{Ch}(S_1) \times_{\text{Ch}(P \times I)} \text{Ch}(S_2) \) \( \implies Z_{cl}(S_1 \bigcup_{P \times I} S_2) = Z_{cl}(S_1) \boxtimes_{Z_{cl}(P \times I)} Z_{cl}(S_2) \)
  - Normalization: \( \text{Ch}(D^2) = pt/G \) \( \implies Z(D^2) = \text{Rep}(G) \)

- Ben-Zvi-Francis-Nadler: \( Z_{cl}(X) \) is uniquely determined by these properties.

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A universal quantization

- So, we define a category $\mathcal{Z}(S)$ of “quasi-coherent sheaves” on the “quantum character variety”, by requiring:
  - Functoriality: $i : S_1 \hookrightarrow S_2 \leadsto Z(i) : Z(S_1) \to Z(S_2)$
  - Isotopies: $i, j : S_1 \hookrightarrow S_2$ isotopic $\leadsto Z(i) \cong Z(j)$
  - Excision: $Z(S_1 \bigcup_{P \times I} S_2) = Z(S_1) \boxtimes_{Z(P \times I)} Z(S_2)$
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Rigorous construction uses “factorization homology” of Lurie/Ayala-Francis-Tanaka.

Braid group and mapping class group actions emerge naturally from isotopies of embeddings.

$\mathcal{Z}(S)$ has a distinguished object, the “quantum structure sheaf” and “global sections functor” $\Gamma$.

Theorem (Ben-Zvi-Brochier-J ’15):
- (Hochschild homology, uses Lyubashenko-Majid CoEnd/braided duals)
- Ribbon graph presentation of $S$, recovering AGS algebras.

Theorem (BZBJ ’16):
For $S$ closed surface, $\mathcal{Z}(S)$ is the quantum Hamiltonian reduction of $A_S$-mod, for an explicitly given multiplicative quantum moment map.

Recovers and generalizes Frohman-Gelca:
- Adding “mirabolic”/Ruijenars-Snijder marked point, Type A spherical double affine Hecke algebras.

$\Rightarrow$ (Balagovic-J ’16)
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- **Theorem** (Ben-Zvi-Brochier-J ‘15): This recovers and generalizes the AGS algebras.
  - $Z(Ann) = \mathcal{O}_q^{RE}(G) / \text{mod}_{\text{Rep}_q(G)}$ (Hochschild homology, uses Lyubashenko-Majid CoEnd/braided duals)
  - Ribbon graph presentation of $S \leadsto Z(S) \simeq A_S / \text{mod}_{\text{Rep}_q(G)}$, recovering AGS algebras.
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- So, we define a category $Z(S)$ of “quasi-coherent sheaves” on the “quantum character variety”, by requiring:
  - Functoriality: $i : S_1 \hookrightarrow S_2 \rightsquigarrow Z(i) : Z(S_1) \rightarrow Z(S_2)$
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- Recovers and generalizes Frohman-Gelca: $\text{End}_{Z(T^2)}(\mathcal{O}_S) = \mathcal{D}_q(H)^W$
- Adding “mirabolic”/Ruijenars-Snijder marked point $\mapsto$ Type A spherical double affine Hecke algebras.
  (Balagovic-J ‘16)
Module structures on $\mathbb{Z}(S)$

- Choice of disk in boundary of $S \leadsto \text{Rep}_q(G)$-module structure on $\mathbb{Z}(S')$
- Get an adjoint pair of module functors: $\text{Rep}_q(G) \rightleftharpoons \mathbb{Z}(S')$
- Choice of boundary component $\mathbb{Z}(\text{Ann})$-action on $\mathbb{Z}(S) \rightarrow$ quantum moment maps.
- Standard techniques ("Barr-Beck") $\rightarrow$ compute $\mathbb{Z}(S)$ recursively using excision.

$Z(\bigodot) = Z(\bigodot) \quad Z(\bigcirc) \quad Z(\bigcirc)$

Quantum Hamiltonian Reduction $\leadsto D_q(H)_{\mathcal{W}}$
Theorem: There exist canonical objects in stratified quantum character varieties, and isomorphisms between and the associated FG chart.

Corollary: AGS and FG quantizations coincide ("quantum trace maps", in any type).

\[ G \backslash \mathcal{O}_q(G/N)/T \]

\[ G \backslash \mathcal{O}_q(G/N \times G/N)/(T \times T) \]

\[ \supseteq \text{Rep}_q(T) \]

\[ G \backslash \mathcal{O}_q(G/N \times G/N \times G/N)/(T \times T \times T) \]

\[ \supseteq \mathcal{O}_q(X_\omega), \text{ a quantum torus } x_i x_j = q x_j x_i. \]

Gluing of quantum torus charts
Theorem (J-Le-Schrader-Shapiro ‘19): There exist canonical objects $\mathcal{O}_{\omega,\Delta}$ in stratified quantum character varieties $Z(\tilde{S})$, and isomorphisms between $End(\mathcal{O}_{\omega,\Delta})$ and the associated FG chart.

Corollary: AGS and FG quantizations coincide, upon localizing (quantum cluster embeddings of AGS algebras).
Three-dimensional structures

- Haugseng, Johnson-Freyd-Scheimbauer: Braided tensor categories naturally form $\mathbf{BrTens}$, a 4-category (iterated Segal space)

Objects: braided tensor categories

1-morphisms: A-B-central tensor categories

2-morphisms: A-B-central C-D-bimodules

3-morphisms: bimodule functors

4-morphisms: bimodule natural transformations
Three-dimensional structures

- Cobordism hypothesis: fully extended n-dimensional TFT’s correspond to n-dualizable objects of BrTens.

- **Theorem (Brochier-J-Snyder ‘18):** Rigid braided tensor categories with enough compact projectives are 3-dualizable in BrTens.
  - This includes $\text{Rep}_q(G)$ for $q$ generic (semisimple).
  - Also Lusztig’s divided powers/restricted $\text{Rep}_q^{Lus}(G)$ for $q$ a root of unity (H.H. Andersen).
  - Also modular tensor category obtained from $\text{Rep}_q^{Lus}(G)$ by quotienting negligibles.

- **Theorem (BJS ‘18):** Modular tensor categories are 4-dualizable, and invertible (known to Freed-Teleman and Walker in different language).

- Hence by the Corbordism Hypothesis, we obtain a fully local (a.k.a. fully extended) TFT $Z: \text{Bord}^{3+\varepsilon/4} \to \text{BrTens}$. In other words,
  - To closed 4-manifolds $W$, it assigns numbers $Z(W)$ (in modular case).
  - To closed 3-manifolds $M$, it assigns a vector space $Z(M)$.
  - To closed surfaces $S$, it assigns a category $Z(S)$, the quantum character variety.
  - To the circle it assigns a monoidal category $Z(S^1) = \text{HH}(\text{Rep}_q(G))$
  - To the point it assigns the braided monoidal category $\text{Rep}_q(G)$
Quantum A-polynomial

- Let $K$ be a knot in $S^3$, let $M$ denote the knot complement. Since $\partial M = T^2$, $M$ defines functors,

$$Z_+(M) : Z(T^2) \rightarrow \text{Vect}, \quad Z_-(M) : \text{Vect} \rightarrow Z(T^2).$$

- Have a global sections functor, $\Gamma = \text{Hom}_{Z(T^2)}(\mathcal{O}_S, -) : Z(T^2) \rightarrow D_q(H)^W \mod$

- In other words, from a knot $K$, we get a system of difference equations, which $q$-deforms the classical A-polynomial $\rightarrow$ canonical construction of (some kind of) quantum A-polynomial.

- Note: Colored Jones $J(K)$ is an element of $Z_{\text{WRT}}(T^2)$ obtained in the same way. One can view $J(K)$ as an element of the $q$-difference system $\Gamma(Z_-(M))$. 
Relative field theories, $\mathcal{Z}$, and WRT

- Let $\mathbf{1}$ denote the trivial TFT in $\text{BrTens}$.

- Definition (Freed-Teleman/Gwilliam-Scheimbauer/Fuchs-Schweigert): A **relative field theory** is given by a dualizable morphism $\text{Rep}_q(G)$ to $\mathbf{1}$ in $\text{BrTens}$.

- Any braided tensor category, regarded as a central algebra over itself defines such a relative field theory.

- Expectation (many people): The WRT 3D TFT is a relative field theory relative to the 4D TFT we constructed above.

- Consequence (of exp.): The colored Jones polynomial $J(K)$ is naturally an element in the “quantum A-polynomial” system we defined above.

- Consequence (of exp.): Alternative construction of Hennings invariant 3D TQFT.
Skein modules and roots of unity

- **Theorem** (Cooke, 18): The subcategory of compact projective objects of $Z(S)$ is the **skein category**, when $q$ is not a root of unity.
  - Corollary: Skein = FG = AGS (quantum trace maps of Bonahon-Wong)
- **Work in progress** (BJS): The skein category is a full (and proper!) subcategory of $Z(S)$, when $q$ is a root of unity. Uses theory of tilting modules.
- **Theorem** (Ganev-J ’18): The affine quantum character variety $\Gamma(A_S)$ is an **Azumaya** algebra over the classical character variety. Methods:
  - Use quantum Frobenius functor $\text{Rep}(G) \to \text{Rep}_q(G)$.
  - Use functoriality of quantum character variety.
  - Prove that quantum Hamiltonian reduction of Azumaya algebras is Azumaya.
- **Corollary** (combining C, BJS, GJ): Skein algebras at roots of unity are Azumaya algebras over their classical counterparts. (a theorem of Bonahon-Wong for SL$_2$)
Summary

- Character varieties satisfy a simple universal property with respect to embeddings of surfaces.
- Replacing $\text{Rep}(G)$ by $\text{Rep}_q(G)$ in universal property gives universal quantization.
- Making further choices one recovers each of the AGS/FG/Skein module presentation.
- Main tools for computing are excision (topology) and Barr-Beck (rep. theory).
- Gives conceptual explanation for mapping class group symmetry, braid group actions, (certain) cluster transformations.
- Gives extended/fully local 3- or 4-dimensional TFT, possibility to use TFT techniques in studying quantum A-polynomial.
- Subtle interesting behavior at roots of unity – no skein description, but observed Azumaya algebra phenomena.