Spectral Theory and Orthogonal Polynomial Ensembles

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Orthogonal Polynomial Ensembles

Let $\mathrm{d}\mu_n(x) = w_n(x)\mathrm{d}x + \mathrm{d}\mu_{n,\mathrm{sing}}(x)$ be a (probability) measure on $\mathbb{R}$ with $\int |t|^j \mathrm{d}\mu(t) < \infty$ for all $j \in \mathbb{N}$.

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$$\mathrm{d}\mathbb{P}_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{i>j} (\lambda_i - \lambda_j)^2 \mathrm{d}\mu_n(\lambda_1) \cdots \mathrm{d}\mu_n(\lambda_n)$$

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We study a sequence of such probability measures.

- We also allow $\mu_n \equiv \mu$ independent of n.

Example: Unitary Invariant Ensembles of Random Matrices

Consider the probability measure

$$\frac{1}{\widetilde{Z}_n} e^{-ntrV(M)} dM$$

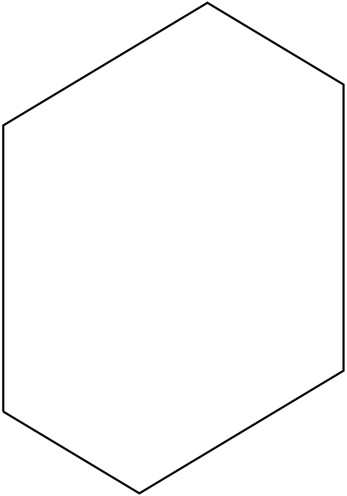
$M \in n \times n$ Hermitian matrices
$V =$ polynomial of even degree with positive leading coefficient.

- Invariant under conjugation by a unitary matrix.
- Induced measure on the n real eigenvalues:

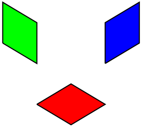
$$\mathrm{d}\mathbb{P}_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{i>j} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{n} e^{-nV(\lambda_j)} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n$$

- GUE: $V(x) = x^2$.

Example: Lozenge tilings



Tilings of the
abc-hexagon with lozenges



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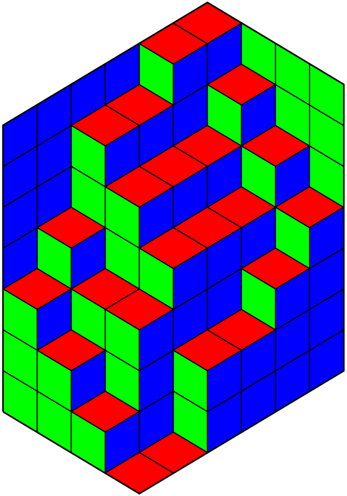

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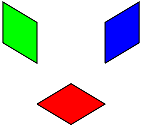

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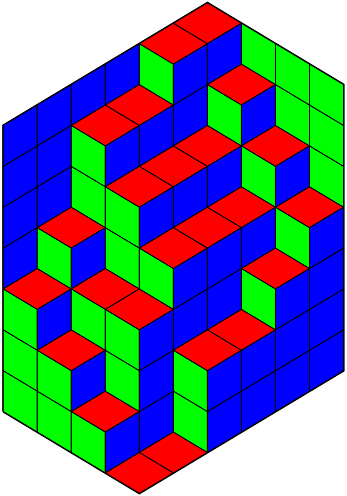

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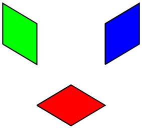

Random tilings: take uniform measure on all possible tilings

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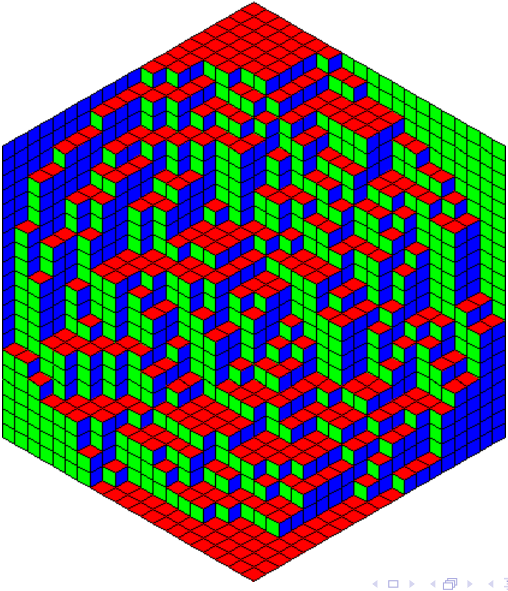

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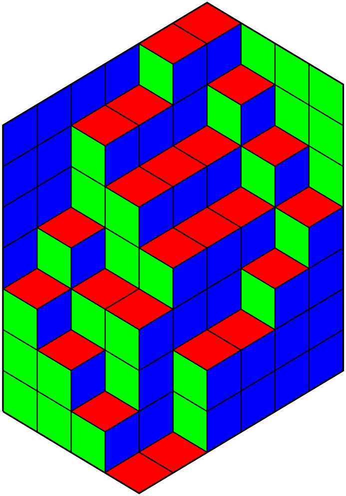

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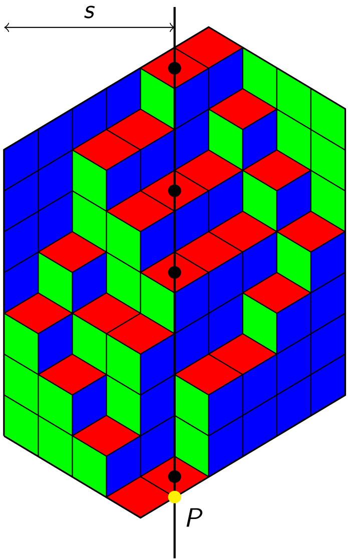

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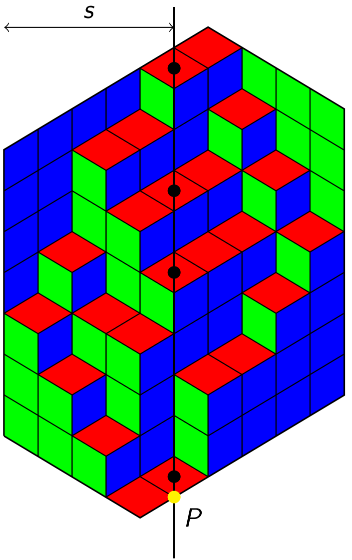

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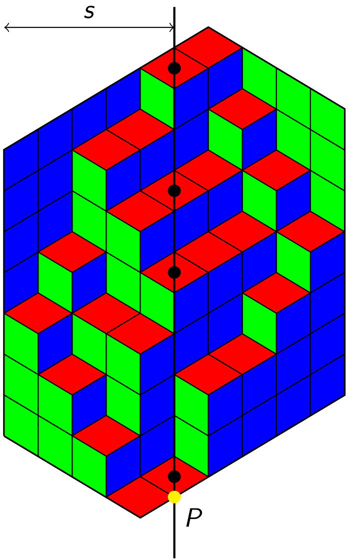

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$\{x_1, \dots, x_{N(s)}\}$ is an orthogonal polynomial ensemble with μ the Hahn weight (Johansson 2002)

Orthogonal Polynomial Ensembles

Other examples (see König '05):

- Random Growth Models (corner-growth model/last passage percolation/TASEP)
- Random Permutations (longest increasing subsequence)
- Nonintersecting Random Walks/Brownian Motions

Orthogonal Polynomial Ensembles

$\prod_{i>j}(\lambda_i - \lambda_j)^2$ = square of a Vandermonde determinant.
By manipulating rows we get

$$\begin{aligned} \mathrm{d}\mathbb{P}_n &= \frac{1}{Z_n}\prod_{i>j}(\lambda_i - \lambda_j)^2 \mathrm{d}\mu(\lambda_1)\cdots\mathrm{d}\mu(\lambda_n) \\ &= \frac{1}{\widetilde{Z}_n}\det\left(K_n(\lambda_i,\lambda_j)_{1\le i,j\le n}\right)\mathrm{d}\mu(\lambda_1)\cdots\mathrm{d}\mu(\lambda_n) \end{aligned}$$

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$$K_n(x, z) = \int K_n(x, y) K_n(y, z) \mathrm{d}\mu(y).$$

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• An orthogonal polynomial ensemble is a *determinantal point process* with kernel $K_n(x,y)$.

Recall

$$\begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}$$

Orthogonal Polynomials and Spectral Theory

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If the moment problem is determinate (in particular if μ has compact support) J is a self-adjoint operator on $\ell^2(\mathbb{N})$ and the spectral measure of the vector $\delta_1 = (1, 0, 0, \dots)^T$ is μ:

$$\left(\delta_1, (J - z)^{-1} \delta_1\right) = \int \frac{\mathrm{d}\mu(t)}{t - z} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Consider the truncation

$$J^{(n)} = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \dots & \dots & \ddots & a_{n-1} \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}.$$

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• How is μ related to asymptotics of $\left\{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\right\}$ and other objects associated with $J^{(n)}$?

Summarizing:

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$$\frac{1}{Z_n} \prod_{i>j} (\lambda_i - \lambda_j)^2 \mathrm{d}\mu(\lambda_1) \cdots \mathrm{d}\mu(\lambda_n) \Longleftarrow \mu$$

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$$\frac{1}{Z_n}\prod_{i>j}(\lambda_i-\lambda_j)^2 \mathrm{d}\mu(\lambda_1)\cdots \mathrm{d}\mu(\lambda_n) \Longleftarrow \mu \Longrightarrow J^{(n)}$$

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- Microscopic Bulk Universality and Clock Spacing.

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Properties of the orthogonal polynomial ensemble $\overset{?}{\leftrightsquigarrow}$ Properties of $J^{(n)}$

- Mean Density and Density of States.
- Microscopic Bulk Universality and Clock Spacing.
- (Local) Law of Large Numbers and Behavior of Truncated Generalized Eigenfunctions (the 'Nevai condition').

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- Mean Density and Density of States.
- Microscopic Bulk Universality and Clock Spacing.
- (Local) Law of Large Numbers and Behavior of Truncated Generalized Eigenfunctions (the 'Nevai condition').
- Central Limit Theorem and Right Limits.

The Mean Density and Density of States

Recall (with $K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)p_j(y)$)

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\widetilde{Z}_n} \det\left(K_n(\lambda_i,\lambda_j)_{1\le i,j\le n}\right) \mathrm{d}\mu(\lambda_1)\cdots\mathrm{d}\mu(\lambda_n).$$

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E.g., for $\mathrm{d}\mu_{\mathrm{n}}(\mathrm{x}) = \mathrm{e}^{-\mathrm{n}\mathrm{V}(\mathrm{x})}\mathrm{d}\mathrm{x}$, ν minimizes

$$\int\int \log\frac{1}{|x-y|}\mathrm{d}\nu(\mathrm{x})\mathrm{d}\nu(\mathrm{y}) + \int \mathrm{V}(\mathrm{x})\mathrm{d}\nu(\mathrm{x})$$

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Theorem (Simon '09, Avron-Simon '83)

For fixed bounded J, the limits of $\frac{K_n(x,x)}{n}\mathrm{d}\mu(x)$ *and* $\mathrm{d}\nu^{(n)}(x)$ *coincide.*

Universality

Assume $\mathrm{d}\nu(x) = \rho(x)\mathrm{d}x$. Fix $x_0 \in \mathrm{Int}\,\mathrm{supp}(\rho)$, a point of continuity for ρ.

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Studying the microscopic behavior at x_0 means studying

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This behavior is conjectured to be 'universal'.

Universality and Clock Spacing of Eigenvalues

Here universality means that

$$\lim_{n\to\infty} \frac{K_n\left(x_0 + \frac{a}{n\rho(x_0)}, x_0 + \frac{b}{n\rho(x_0)}\right)}{K_n(x_0, x_0)} = \frac{\sin \pi(a-b)}{\pi(a-b)}$$

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The Christoffel-Darboux formula says:

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)p_j(y) = a_n \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}$$

So if $p_n(x) = 0$ then $K_n(x,y) = 0$ iff $p_n(y) = 0$.

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Theorem (Freud ('71), Levin-Lubinsky ('08))

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This type of asymptotic behavior is called clock spacing.

Universality

Some background:

• Invariant Ensembles: Dyson, Gaudin and Mehta ('60s); Bleher and Its ('99); Deift, Kriecherbauer, McLaughlin, Venakides and Zhou ('99); Pastur and Shcherbina ('97).

• Discrete Orthogonal Polynomial Ensembles (*discrete* sine kernel): Baik, Kriecherbauer, McLaughlin, Miller ('07).

• Wigner matrices: Erdős, Péché, Ramírez, Schlein, Yau ('10); Tao and Vu ('11).

Universality

Recently, Lubinsky introduced two methods for establishing universality for orthogonal polynomial ensembles with μ locally absolutely continuous:

I. ('09) Comparison inequality: If $\mu \leq \mu^*$ then

$$\left|\frac{K_n(x,y)-K_n^*(x,y)}{K_n(x,x)}\right| \leq \left(\frac{K_n(y,y)}{K_n(x,x)}\right)^{1/2}\left(1-\frac{K_n^*(x,x)}{K_n(x,x)}\right)^{1/2}.$$

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Extensions by Findley ('08), Simon ('08), Totik ('09). Maltsev ('10) applied these ideas to continuum Schrödinger operators.

Universality

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for a, b in fixed compacts.
Thus, this is a normal family and any limit is an entire function of exponential type. By studying properties of the limit it follows that in this case

Universality along the diagonal $\Longleftrightarrow$ Universality.

Universality

• This approach was applied to varying measures by Levin and Lubinsky ('08), and also to show universality at the hard (Lubinsky '08) and soft edges of the spectrum (Levin-Lubinsky '10). It was also used by Avila, Last and Simon ('10) to show that universality holds in the absolutely continuous spectrum for ergodic Jacobi matrices (where the spectrum could also be a Cantor set).

Universality and Clock

Summarizing:

Local absolute continuity of μ (+Other conditions) $\Rightarrow$ Universality $\Rightarrow$ Clock.

Universality and Clock Spacing of Eigenvalues

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Theorem (B ('11))

There exist Jacobi matrices whose spectral measures are purely singular continuous on $[-2, 2]$ such that universality, and hence also clock behavior, holds for all $x_0 \in (-2, 2)$.

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The examples are Schrödinger operators with a sparse potential!

• Universality $\Longrightarrow$ lower bound on the spectral measure? (B, Last, Simon '14 – partial results).

Fluctuations on a Large Scale

The linear statistic associated with a function $f : \mathbb{R} \to \mathbb{R}$, is

$$X_f^{(n)} = \sum_{j=1}^{n} f(\lambda_j),$$

where $(\lambda_1, \lambda_2, \dots, \lambda_n)$ is random from the orthogonal polynomial ensemble.

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We want to study asymptotics of $X_f^{(n)} - \mathbb{E}X_f^{(n)}$.

The determinantal structure means that

$$\mathbb{E}X_f^{(n)} = \int f(x) K_n(x,x) \mathrm{d}\mu(x),$$

$$\mathrm{Var} X_f^{(n)} = \int\int f(x)^2 K_n(x,x) \mathrm{d}\mu(x) - \int\int f(x) f(y) K_n(x,y)^2 \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

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The Christoffel-Darboux formula says:

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(x)p_j(y) = a_n \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x - y}$$

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So

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&= \frac{1}{2}\int\int \left(f(x)-f(y)\right)^2 K_n(x,y)^2 \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
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$$= \frac{a_n^2}{2}\int\int \frac{\left(f(x)-f(y)\right)^2}{(x-y)^2}\left(p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)\right)^2 \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

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\end{aligned}$$

$$\leq a_n^2 \sup_{x,y}\left|\frac{f(x)-f(y)}{x-y}\right|^2$$

Fluctuations on a Large Scale

Thus if a_n is bounded, $\mathrm{Var} X_f^{(n)}$ is bounded for f Lipschitz!

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We expect

- A Law of Large Numbers:

$$\frac{1}{n}\left(X_f^{(n)} - \mathbb{E}X_f^{(n)}\right) \to 0.$$

- A Central Limit Theorem:

$$X_f^{(n)} - \mathbb{E}X_f^{(n)} \to N\left(0, \sigma(f)^2\right).$$

A Law of Large Numbers

Law of Large Numbers

Theorem (Global Law of Large numbers for OPE, B-Duits)

There exists a universal constant, $A > 0$, such that for any measure with finite moments, μ, any bounded function, f, and any $\varepsilon > 0$

$$\mathbb{P}\left(\frac{1}{n}\left|X_f^{(n)} - \mathbb{E}X_f^{(n)}\right| \geq \varepsilon\right) \leq 2\exp\left(-n\varepsilon \min\left(\frac{\varepsilon}{8A\|f\|_\infty^2}, \frac{1}{6\|f\|_\infty}\right)\right)$$

for all $n \in \mathbb{N}$.

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In particular, if μ_n is a sequence of such measures for which also $\frac{K_n(x,x)}{n}\mathrm{d}\mu_n(x)$ has a weak limit, ν, and f is bounded and continuous, then

$$\lim_{n\to\infty} \frac{1}{n} X_f^{(n)} = \int f(x)\ \mathrm{d}\nu(x),$$

almost surely.

Law of Large Numbers

• In case $\frac{\mathbb{E}X_f^{(n)}}{n}$ has no limit, there is still a law of large numbers along subsequences where there is a limit.

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• Ingredients:

A general large deviation estimate for determinantal processes with kernels that are self-adjoint projections

$$+$$

A bound on the variance of $X_f^{(n)}$

A General Large Deviation Estimate

For any determinantal point process with correlation kernel that is a finite rank self-adjoint projection we show

$$\mathbb{P}(|X_f - \mathbb{E}X_f| \geq \varepsilon) \leq \begin{cases} 2\exp\left(-\frac{\varepsilon^2}{4A\mathrm{Var}X_f}\right), & \text{if } \varepsilon < \frac{2A\mathrm{Var}X_f}{3\|f\|_\infty} \\ 2\exp\left(-\frac{\varepsilon}{6\|f\|_\infty}\right), & \text{if } \varepsilon \geq \frac{2A\mathrm{Var}X_f}{3\|f\|_\infty} \end{cases}$$

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The concentration inequality is uniform in the measure μ. The constant $A > 0$ that we derive in the proof is

$$A = 2\mathrm{e}^2 \sum_{m=0}^{\infty} (\mathrm{e}/3)^m (m+2)^{3/2}.$$

A Bound on the Variance

Recall

$$\frac{\mathbb{E}X_f^{(n)}}{n} = \int f(x)\frac{K_n(x,x)}{n}\mathrm{d}\mu(x).$$

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So

$$\frac{\mathrm{Var}\ X_f^{(n)}}{n} = \int\left(\int (f(x)-f(y))\frac{K_n(x,y)^2}{K_n(x,x)}\mathrm{d}\mu(y)\right)f(x)\frac{K_n(x,x)}{n}\mathrm{d}\mu(x).$$

The Nevai Condition

A measure, μ, is said to satisfy the Nevai condition at x if for any continuous, compactly supported function, f,

$$\int (f(y) - f(x)) \frac{K_n(x,y)^2}{K_n(x,x)} \mathrm{d}\mu(y) \to 0,$$

as $n \to \infty$.

I.e. if

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Thus:

$$\text{(Uniform Nevai condition)} + \left(\frac{K_n(x,x)}{n} \text{ bounded on } \mathsf{supp}(f) \right) \implies \text{Var } X_f = o(n).$$

The Nevai Condition

• Some history: Nevai ('79); Nevai ('86); Criscuolo, Mastroianni and Nevai ('89); Nevai, Totik and Zhang ('91); Lubinsky and Nevai ('92); Zhang ('93); Szwarc ('95); Della Vecchia ('02); Lasser and Obermaier ('03); B, Last and Simon ('10); Lubinsky ('11).

The Nevai Condition–Spectral Interpretation

Theorem (B-Last-Simon '10)

For fixed μ *with compact support s.t.* $\inf(a_n) > 0$, *the Nevai condition at* x *is equivalent to*

$$\lim_{n\to\infty} a_n^2 \frac{|p_n(x)|^2}{\sum_{j=0}^{n-1} |p_j(x)|^2} = 0.$$

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Recall

$$J \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}.$$

The Nevai Condition–Spectral Interpretation

Letting

$$J^{(n)} = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \dots & \dots & \ddots & a_{n-1} \\ 0 & \dots & \dots & a_{n-1} & b_n \end{pmatrix}$$

The Nevai Condition–Spectral Interpretation

we get

$$\frac{\left\| \left(J^{(n)} - x\right) \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \end{pmatrix} \right\|^2}{\left\| \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \end{pmatrix} \right\|^2} = a_n^2 \frac{|p_n(x)|^2}{\sum_{j=0}^{n-1} |p_j(x)|^2}$$

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Nevai condition at $x \Longrightarrow$ truncated generalized eigenfunction is an approximate eigenfunction at x.

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- This is known in many particular cases (see the list above).
- B-Duits show that universality at x+absolute continuity $\Longrightarrow$ Nevai condition holds at x.

A Local Law of Large Numbers

For $\alpha \in (0,1)$ and x^*, let

$$X_{f,\alpha,x^*}^{(n)} = \sum_{j=1}^{n} f\left(n^{\alpha}(\lambda_j - x^*)\right)$$

The proof of the following *local* LLN uses a local version of the Nevai condition.

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The proof of the following *local* LLN uses a local version of the Nevai condition.

Theorem (Local Law of Large Numbers for OPE, B-Duits '13)

Assume μ is regular and assume that on a neighborhood of x^*, μ is absolutely continuous with continuous and nonvanishing derivative. Then for any continuous f with compact support and $\varepsilon > 0$

$$\mathbb{P}\left(n^{\alpha}\left|\frac{X^{(n)}_f}{n} - \frac{\mathbb{E}X^{(n)}_f}{n}\right| \geq \varepsilon\right) \leq 2\exp\left(-\varepsilon n^{1-\alpha}/6\|f\|_\infty\right).$$

A Central Limit Theorem

Central Limit Theorem

Theorem (B-Duits '13)

Assume that for any $\ell \in \mathbb{Z}$,

$$a_{n+\ell,n} \to a \qquad b_{n+\ell,n} \to b.$$

Then for any polynomial with real coefficients, f,

$$X_f^{(n)} - \mathbb{E}X_f^{(n)} \to N\left(0, \sum_{k=1}^{\infty} k|\widehat{f}_k|^2\right)$$

where

$$\widehat{f}_k = \frac{1}{2\pi}\int_0^{\infty} f\,(2a\cos\theta + b)\, e^{-i\theta k}\mathrm{d}\theta$$

Central Limit Theorem for C^1 functions

Theorem (B-Duits 13)

Suppose there is a compact $E \subset \mathbb{R}$ such that for any $k \in \mathbb{N}$ we have

$$\int_{\mathbb{R}\setminus E} |x|^k K_n(x,x)\mathrm{d}\mu(x) = o(1/n),$$

as $n \to \infty$. Then the CLT holds for any $f \in C^1(\mathbb{R})$ such that $|f(x)| \leq C(1+|x|^k)$ for some $C > 0$ and $k \in \mathbb{N}$.

Corollary

If μ is non-varying and has compact support then the CLT holds for any $f \in C^1(\mathrm{supp}(\mu))$.

Examples

- Unitary Ensembles.

The eigenvalues are an orthogonal polynomial ensembles with

$$\mathrm{d}\mu_n(x) = \mathrm{e}^{-nV(x)}\mathrm{d}x.$$

with $\mathrm{supp}(\nu) = [\gamma, \delta]$.
A Central Limit Theorem in this case has been proven by Johansson ('98) for the case that $\mathrm{supp}(\nu)$ is a single interval. His techniques and results were extended by Kriecherbauer and Shcherbina ('10) and Borot and Guionnet ('12).

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The theorem above reproduces these results.

Examples

- Lozenge tiling of the hexagon.

The locations of  on a vertical section are an orthogonal polynomial ensemble with

$$\mathrm{d}\mu = \sum_{x=0}^{N} \binom{\alpha + x}{x} \binom{\beta + N - x}{N - x} \delta_x .$$

This is called the *Hahn* weight and the orthogonal polynomials are the Hahn polynomials.
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This is called the *Hahn* weight and the orthogonal polynomials are the Hahn polynomials.
A 'two-dimensional' Central Limit Theorem in this case has been proven by Petrov ('13)
A Central Limit Theorem for the one-dimensional sections also follows from the theorem above.

Examples

The following result for the non-varying situation follows by combining the CLT with the celebrated Denisov-Rakhmanov Theorem.

Theorem

Fix

$$\mathrm{d}\mu = w(x)\mathrm{d}x + \mathrm{d}\mu_{sing},$$

and assume that $\operatorname{supp}_{ess}(\mu) = [\gamma, \delta]$ *and* $w(x) > 0$ *Lebesgue a.e. on* $[\gamma, \delta]$*. Then for any* $f \in C^1(\mathbb{R})$ *we have*

$$X_f^{(n)} - \mathbb{E}X_f^{(n)} \to N\left(0, \sum_{k=1}^{\infty} k|f_k|^2\right)$$

where

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{\delta-\gamma}{2}\cos\theta + \frac{\delta+\gamma}{2}\right) e^{-\mathrm{i}k\theta}\mathrm{d}\theta.$$

Central Limit Theorem

A CLT holds also along subsequences:

Theorem (B-Duits '13)

Assume that there exists a subsequence $\{n_j\}_j$ *such that for any* $\ell \in \mathbb{Z}$ *we have*

$$a_{n_j+\ell, n_j} \to a \qquad b_{n_j+\ell, n_j} \to b$$

as $j \to \infty$. *Then for any polynomial with real coefficients,* f,

$$X_f^{(n_j)} - \mathbb{E} X_f^{(n_j)} \to N\left(0, \sum_{k=1}^{\infty} k |\hat{f}_k|^2\right)$$

where

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(2a\cos\theta + b) e^{-\mathrm{i}\theta k} \mathrm{d}\theta.$$

Example

Thus, it is possible that for a fixed measure μ we have different CLT's for different subsequences:

By taking Jacobi parameters $b_n \equiv 0$ and a_n varying at an increasingly slower rate between $1/2$ and 1 we can have that for each $a \in [1/2, 1]$ there is a subsequence $n_j(a)$ such that for each fixed $\ell \in \mathbb{Z}$.

$$a_{n_j(a)+k} \to a, \qquad \text{as } j \to \infty.$$

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In this example a Law of Large Numbers still holds:

$$\lim_{n\to\infty} \frac{1}{n} X_n(f) = \int f(x) \mathrm{d}\nu_{eq}(x)$$

where ν_{eq} is the equilibrium measure for the interval $[-2,2]$.

Right Limits

The general result involves the notion of *right limit* for Jacobi matrices:

Suppose $\{J_n\}_n$ is a sequence of one-sided Jacobi matrices.

Definition

J^R, a two-sided Jacobi matrix, is called a right limit of $\{J_n\}_n$ if for some $n_j \to \infty$ and all $k, \ell \in \mathbb{Z}$ (but *not* necessarily uniformly in k, ℓ),

$$\left(J^R\right)_{k,\ell} = \lim_{j\to\infty} \left(J_{n_j}\right)_{n_j+k,n_j+\ell}$$

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• Right limits of Jacobi matrices and Schrödinger operators were introduced as a tool in spectral theory by Last and Simon ('99), and have been used to study absolutely continuous spectrum (Last-Simon '99, Remling '11), essential spectrum (Last-Simon '06), and even boundary behavior of power series (B-Simon '11).

Examples

- If for any $k \in \mathbb{Z}$

$$a_{n+k,n} \to a \qquad b_{n+k,n} \to b$$

then the unique right limit of J is the Laurent matrix L with

$$L_{i,j} = \begin{cases} a & |i-j| = 1 \\ b & i = j \\ 0 & \text{otherwise} \end{cases}$$

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- Let $J_n \equiv J$ with parameters a_j, b_j. Then

$$a_n \longrightarrow a \qquad b_n \longrightarrow b$$

iff L as above is the unique right limit of J.

CLT from Right Limits

Let $s(z) = \sum_{j=-q}^{p} s_j z^j$ be a Laurent polynomial. We denote by $L(s)$ the associated Laurent matrix:

$$(L(s))_{i,j} = s_{i-j}.$$

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Theorem

Suppose $L(s)$ is a right limit of $\{J_n\}_n$ along the subsequence n_j. Then for any polynomial with real coefficients, f,

$$X_f^{(n_j)} - \mathbb{E}X_f^{(n_j)} \to N\left(0, \sum_{k=1}^{\infty} k|\widehat{f_k}|^2\right),$$

where

$$\widehat{f_k} = \frac{1}{2\pi i}\oint_{|z|=1} f(s(z)) \frac{dz}{z^{k+1}}$$

A General Limit Theorem

Theorem

Let J^R be a right limit of J (bounded and fixed!) with subsequence $\{n_j\}_j$. Define

$$\left(J_M^R\right)_{kl} = \begin{cases} (J^R)_{kl}, & k,l=-M,\ldots,M, \\ 0, \ otherwise \end{cases}.$$

Let P_- be the projection on the negative coefficients

$(P_- x)_k = \begin{cases} x_k & k<0 \\ 0 & k\geq 0. \end{cases}$. *Then for any polynomial f we have*

$$\lim_{j\to\infty} \mathbb{E}\left[\exp t(X_f^{(n_j)} - \mathbb{E}X_f^{(n_j)})\right]$$
$$= \lim_{M\to\infty} \mathrm{e}^{t\, TrP_- f(J_M^R)} \det\left(I + P_-(\mathrm{e}^{tf(J_M^R)} - I)P_-\right)$$

In particular, both limits exist.

A General Limit Theorem

• It is an interesting open problem to find the limiting value at the right-hand side for various possible J^R. For example, if J^R is two-periodic this should of course match with known formulas for the multi-cut(=finite gap) case (see Shcherbina '13, Borot-Guionnet '13).

A General Limit Theorem

- It is an interesting open problem to find the limiting value at the right-hand side for various possible J^R. For example, if J^R is two-periodic this should of course match with known formulas for the multi-cut(=finite gap) case (see Shcherbina '13, Borot-Guionnet '13).

- While in the general finite gap case a CLT does not hold for any f in general, in the case that the related Jacobi matrix is periodic, a CLT does hold for $f \circ \Delta$ where Δ is the associated discriminant!

A Word About the Proofs

Cumulant Expansion

Expand the moment-generating function

$$\begin{aligned} \mathbb{E}\left[\exp tX_f^{(n)}\right]) &= \det(1 + (\mathrm{e}^{tf} - 1)K_n) \\ &= \exp \mathrm{Tr} \log \left((\mathrm{e}^{tf} - 1)K_n\right) \\ &= \exp\left(\sum_{m=1}^{\infty} t^m C_m^{(n)}(f)\right) \end{aligned}$$

where

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i\geq 1} \frac{\mathrm{Tr} f^{l_1} K_n \cdots f^{l_j} K_n}{l_1! \cdots l_j!}$$

are the cumulants.

Estimating the Cumulants

For a Central Limit Theorem one needs to show that

$$\lim_{n\to\infty} C_m^{(n)}(f) \to \begin{cases} 2\sigma(f)^2, & m = 2 \\ 0, & m \geq 3. \end{cases}$$

Problem: each term in the sum

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i\geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n}{l_1! \cdots l_j!}$$

grows linearly with n. Some effective cancellation occurs!

Estimating the Cumulants

By using the identity

$$\sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{1}{l_1! \cdots l_j!} = 0$$

for $m \geq 2$, we may write

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n}{l_1! \cdots l_j!}$$

for $m \geq 2$.

Now each term in the double sum can be shown to be bounded (effectively by the variance). This captures a first cancellation.

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This is the first key step towards the concentration inequalities!

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for $m \geq 2$.

Now each term in the double sum can be shown to be bounded (effectively by the variance). This captures a first cancellation.

This is the first key step towards the concentration inequalities!

For a CLT we need to identify further cancellations.

Estimating the Cumulants

Use orthogonality to rewrite

$$C_m^{(n)}(f) = \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f^{l_1} K_n \cdots f^{l_j} K_n - \operatorname{Tr} f^m K_n}{l_1! \cdots l_j!}$$

$$= \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{l_1+\cdots+l_j=m, l_i \geq 1} \frac{\operatorname{Tr} f(J)^{l_1} P_n \cdots f(J)^{l_j} P_n - \operatorname{Tr} f(J)^m P_n}{l_1! \cdots l_j!}$$

By using the band structure of J (and hence of $f(J)$ for polynomial f) it is not hard to prove that

$$\operatorname{Tr} f(J)^{l_1} P_n \cdots f(J)^{l_j} P_n - \operatorname{Tr} f(J)^m P_n$$

only depends on a finite and fixed number of recurrence coefficients.

Thus only a relatively small part of J matters in the fluctuation.

A Comparison Principle

This leads to the following principle:

Comparison/Universality principle

If J_1 and J_2 have the same right limit J^R along the same subsequence $\{n_j\}$, then the corresponding cumulants have the same limits along these subsequences.

Hence if J has a Laurent matrix $L(s)$ as a right limit, then we can assume that J is a matrix with constant diagonals from the start!

It follows that we only need to compute the CLT for this special case.

A CLT for Toeplitz Matrices

Lemma (B-Duits 13)

Let $s(z) = \sum_{j=-q}^{p} s_j z^j$ be a Laurent polynomial. Then for $T(s)$, the associated Toeplitz matrix,

$$\lim_{n\to\infty} \det\left(I + P_n(\mathrm{e}^{T(s)} - I)P_n\right)\mathrm{e}^{-\operatorname{Tr} P_n T(s)} = \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} k s_k s_{-k}\right)$$

The proof is based on Ehrhardt's generalization of the Helton-Howe-Pincus formula: if $[A, B]$ is trace class, then

$$\det \mathrm{e}^{-A}\mathrm{e}^{A+B}\mathrm{e}^{-B} = \exp\frac{1}{2}\operatorname{Tr}[A, B].$$

(This formula essentially captures the final cancellations in the cumulant expansion)

Outlook

• Extensions: The CLT described can be extended to biorthogonal ensembles with a recurrence (e.g. the two matrix model).

• $\beta \neq 2$.

• Mesoscopic CLT.

• Fine spectral properties of Jacobi matrices.

Thank you for your attention