Spectral Theory of Orthogonal Polynomials

Barry Simon

IBM Professor of Mathematics and Theoretical Physics
California Institute of Technology
Pasadena, CA, U.S.A.

Lecture 8: Finite Gap Isospectral Torus

- Lecture 6: Szegő Asymptotics and Shohat-Nevai for $[-2,2]$
- Lecture 7: Periodic OPRL
- Lecture 8: Finite Gap Isospectral Torus
- Lecture 9: Fuchsian Groups and Finite Gaps, I

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Overview

We've seen that whole line periodic Jacobi matrices lead to band spectrum, $\cup_{j=1}^{\ell+1}[\alpha_j, \beta_j]$. Here ℓ will indicate the number of gaps and, at least at the start, we suppose all gaps are open. Recall that the spectrum is determined by the discriminant, Δ_J, via $\sigma(J) = \Delta^{-1}\big([-2,2]\big)$.

Conversely, as we saw, Δ_J is determined by $\sigma(J)$ so

$$\sigma(J) = \sigma(J') \Leftrightarrow \Delta_J = \Delta_{J'}$$

In this section, we'll explore when two J's have the same spectrum, not only in the periodic case but for general finite gap sets. We'll see this "isospectral manifold" is an ℓ-dimensional torus.

Quadratic Irrationalities

In the latter half of the 18th century, Euler and Legendre discovered that a numeric canonical continued fraction has periodic coefficients if and only if its value x obeyed a quadratic equation. We know that Jacobi parameters are coefficients in the continued fraction expansion of a half-line m function. Thus, we expect periodic Jacobi parameters should be connected to m-functions obeying a quadratic equation.
In the periodic case, we have

[Note: $-a_p u_{p-1} \binom{m}{-1} = \binom{u_1}{a_p u_0}$ with u = Weyl solution]

$$T_p(z) \begin{pmatrix} m \\ -1 \end{pmatrix} = c \begin{pmatrix} m \\ -1 \end{pmatrix}, T_p(z) = \begin{pmatrix} p_p & -q_p \\ a_p p_{p-1} & -a_p q_{p-1} \end{pmatrix}$$

Thus:

$$m = -\frac{mp_p + q_p}{a_p(mp_{p-1} + q_{p-1})}$$

$$\alpha(z)m(z)^2 + \beta(z)m(z) + \gamma(z) = 0$$

$$\alpha(z) = a_p p_{p-1}(z), \quad \beta(z) = p_p(z) + a_p q_{p-1}(z),$$

$$\gamma(z) = q_p(z).$$

$$\beta^2 - 4\alpha\gamma = (p_p - a_p q_{p-1})^2 - 4[a_p(q_p p_{p-1} - p_p q_{p-1})]$$

$$= \Delta^2 - 4 \text{ where } \Delta = p_p - a q_{p-1}$$

is our old friend, the discriminant (bad name!).

Riemann Surface of $m(z)$

Thus $m(z) = \dfrac{-\beta(z) \pm \sqrt{\Delta^2 - 4}}{2\alpha(z)}$; $\deg \Delta = p$, $\deg \alpha = p - 1$, $\deg \beta = p$.

If all gaps are open, $\Delta^2 - 4$ has a square root singularity exactly at $\{\alpha_j, \beta_j\}_{j=1}^{p}$, i.e., edges of bands. The natural branch cuts are the bands $\cup[\alpha_j, \beta_j]$.

$m(z)$ is meromorphic (i.e., analytic from $\mathcal{S}$ to $\mathbb{C} \cup \{\infty\}$) on $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$, two copies of $\mathbb{C} \cup \{\infty\} \cup_{j=1}^{p} [\alpha_j, \beta_j)$ glued at the bands.

$\pi : \mathcal{S} \to \mathbb{C} \cup \{\infty\}$ maps a point in $\mathcal{S}$ to underlying $\mathbb{C}$.

The genus of $\mathcal{S}$ is ℓ; it is a sphere with ℓ handles.

Riemann Surface of $m(z)$

The Riemann surface can be realized as the set of points (z, w) in $\mathbb{C}^2$ with $(\pi((z, w)) = z)$

$S(z, w) = w^2 - \big(\Delta(z)^2 - 4\big) = 0$

$\nabla S = (2w, 2\Delta(z)\Delta'(z))$

Since no gaps are closed, if $w = 0$, $\Delta(z) \neq 0 \neq \Delta'(z)$, so $\mathcal{S}$ is a complex manifold. At points with $w \neq 0$, $\frac{\partial S}{\partial w} \neq 0$ and so w can be written as a function of z, i.e., z is a local coordinate.

When $w = 0$, i.e., $\Delta = \pm 2$, $\frac{\partial S}{\partial z} \neq 0$ and z can be written as a function of $w = (z - z_0)^{1/2} + O\big((z - z_0)\big)$. That is at branch points, Riemann surface coordinates are $(z - z_0)^{1/2}$.

Poles of $m(z)$

Near ∞_+, $\sqrt{\Delta^2 - 4} = \Delta\left(\sqrt{1 - 4\Delta^{-2}}\right) = \Delta + O(\frac{1}{\Delta})$.

$-\beta + \sqrt{\Delta - 4^2} \sim -(p_p + a_p q_{p-1}) + (p_p - a_p q_{p-1}) = 2a_p q_{p-1} = O(z^{p-2})$, while $\alpha(z) = 2a_p p_{p-1}(z) = O(z^{p-1})$.

Thus, near ∞_+, $m(z) \to 0$ as z^{-1} (as it must, as an m-function).

Near ∞_-, $\sqrt{\Delta^2 - 4}$ has opposite sign and numerator is $\sim -2\beta = O(z^p)$, so $m(z)$ has a simple pole at ∞_-.

All other possible poles are at zeros of α.

Poles of $m(z)$

As we saw, near ∞, $m(z) = O(1/z)$ because $\sqrt{\Delta^2 - 4} \sim \Delta = (a_1 \cdots a_p)^{-1} z^p + O(z^{p-1})$, so $\sqrt{\Delta^2 - 4}$ is positive on (β_p, ∞).

$\Delta^2 - 4$ has a simple zero at β_p, so $\arg\left(\sqrt{\Delta^2 - 4}\right) = \frac{\pi}{2}$ on (α_p, β_p) consistent with $\operatorname{Im} m \geq 0$ there so long as α has no zeros in (α_p, ∞) (since $\alpha(x) > 0$ near $+\infty$ on $\mathbb{R}$).

Since there are simple zeros of $\Delta^2 - 4$ at α_p and β_{p-1}, $\arg\left(\sqrt{\Delta^2 - 4}\right) = \pi$ on (β_{p-1}, α_p) and $\arg\left(\sqrt{\Delta^2 - 4}\right) = \frac{3\pi}{2}$.

For $\operatorname{Im} m \geq 0$ on $(\alpha_{p-1}, \beta_{p-1})$, α must be negative there, i.e., α has an odd number of zeros in $[\beta_{p-1}, \alpha_p]$.

Poles of $m(z)$

We thus see $\alpha(z)$ has at least one zero in each gap but $\deg\alpha = p - 1$ and there are $p - 1$ gaps. We have thus proven

Theorem. *α has exactly one zero in the closure of each gap and no other zeros.*

Poles of $m(z)$

When $\alpha = 0$, $\Delta^2 - 4 = \beta^2 - 4\alpha\gamma = \beta^2$, so numerator is $-\beta \pm \beta$ (or $-\beta \mp \beta$), so if $\alpha(x_0) = 0$ and $x_0 \in (\beta_{j-1}, \alpha_j)$ then $\beta^2 = \Delta^2 - 4 \neq 0$ and we get a pole at exactly one point with $\pi(z) = x_0$.

If $x_0 \in \{\beta_{p-1}, \alpha_j\}$, then $\alpha = O(x - x_0)$ while $-\beta(x) + \sqrt{\Delta^2 - 4}$ is $O\big((x - x_0)\big)^{1/2}$, so $m \sim (z - z_0)^{-1/2}$.

Since $(z - z_0)^{1/2}$ is local coordinate at the branch point, still a simple pole.

Theorem. Suppose all gaps are open. On $\mathcal{S}$, $m(z)$ has exactly p poles, all simple; one at ∞_- and exactly one in each $\pi^{-1}\big([\beta_{j-1}, \alpha_j]\big)$, $j = 2, \dots, p$.

Periodic Isospectral Torus

Each $\pi^{-1}\big([\beta_j, \alpha_j]\big)$ is two copies of (β_{j-1}, α_j) glued at ends, i.e. a circle.

The set of possible poles of m is $(\partial\mathbb{D})^\ell$, i.e., a torus.

We claim the map of $m \mapsto$ poles is a bijection of the isospectral manifold and an ℓ-dimensional torus.

Let's try to construct an m given a set of possible poles.

Periodic Isospectral Torus

For let $R = \prod_{j=1}^{p}(z-\alpha_j)(z-\beta_j)$. We want to try

$$m(z) = \frac{-B(z) + \sqrt{R(z)}}{A(z)}$$

with $\deg B = p$, $\deg A = p-1$.

Near ∞_+, $\sqrt{R}$ has a Taylor expansion $z^p + Cz^{p-1} + \dots$

For $-B(z) + \sqrt{R(z)}$ to be $O(z^{p-2})$ at ∞_+, we know $B(z) = z^p + Cz^{p-1} + \dots$.

Let $\sqrt{R(p_j)}$ be the value of $\sqrt{R}$ at the poles p_j with $\pi(p_j) \in [\alpha_j, \beta_{j+1}]$. We need $B\big(\pi(p_j)\big) = -\sqrt{R(p_j)}$. This gives $p-1$ additional pieces of data and determines B.

Periodic Isospectral Torus

$A(z) = C\prod_{j=1}^{p-1}(z - \pi(p_j))$. C is determined by $m(z) = -z^{-1} + O(z^{-2})$ near ∞_+.

By the analysis of $\arg(\sqrt{R})$ as above, m constructed this way has $\operatorname{Im} m(x + i0) > 0$ on each $[\alpha_j, \beta_j]$ in $\mathcal{S}_+$. Poles and residues which are in $\mathcal{S}_+$ determine point mass of $d\mu$. Thus, we get measure μ and so isospectral m-function with that pole data.

Periodic Isospectral Torus

One needs a little more analysis to confirm that m has periodic Jacobi parameters. The result is

Theorem. *The map of half-line J's of period p with a given Δ is mapped bijectively to $(\partial\mathbb{D})^{p-1}$ by taking $J \mapsto m \mapsto$ poles in $\mathcal{S}$.*

If a gap is closed, the torus shrinks to one lower dimension.

General Finite Gap Set

The above analysis works for any finite gap set that is given

$\alpha_1 < \beta_1 < \alpha_2 < \ldots < \alpha_{\ell+1} < \beta_\ell$ in $\mathbb{R}$

one can form $R = \prod_{j=1}^{\ell+1}(z-\alpha_j)(z-\beta_j)$

and the Riemann surface of $\sqrt{R}$ formed by gluing $\mathcal{S}_+$ and $\mathcal{S}_-$ together

to get $\mathcal{S}$ with points at ∞.

General Finite Gap Set

$\mathcal{S}$ has genus ℓ and $\pi : \mathcal{S} \to \mathbb{C} \cup \{\infty\}$. We also define $\tau : \mathcal{S} \to \mathcal{S}$ which takes any $z \in \mathcal{S}_+$ to the unique $\tau(z)$ in $\mathcal{S}_-$ with $\pi(z) = \pi\big(\tau(z)\big)$.

Meromorphic functions are "analytic" maps $f : \mathcal{S} \to \mathbb{C} \cup \{\infty\}$, the Riemann sphere.

By the general theory, any such f has a degree, d, i.e., a number so that for any $w \in \mathbb{C} \cup \{\infty\}$, $f(z) - w$ has d roots counting multiplicity.

General Finite Gap Set

$\mathcal{S}$ is hyperelliptic—namely there exists functions of degree 2—any function of the form $f\big(\pi(z)\big)$ where f is an analytic bijection of $\mathbb{C} \cup \{\infty\}$ to itself.

Such functions obey $f\big(\tau(z)\big) = f(z)$. If that fails, we say that f is not square-root free.

The minimal degree of not square-root free functions is $\ell + 1$ (e.g., $\sqrt{R}$).

General Finite Gap Set

As an analysis like the periodic case shows, there is a one-one correspondence between minimal degree functions $m(z)$ with $\operatorname{Im} m(z) > 0$ if $z \in \mathcal{S}_+ \cap \mathbb{C}_+$ with $m(z) = -z^{-1} + O(z^{-2})$ at ∞_+ and with poles at ∞_- and on $\mathbb{R} \cap \mathcal{S}$ ("minimal Herglotz functions") and the ℓ-dimensional torus of

$$\bigtimes_{j=1}^{\ell} \pi^{-1}\big([\beta_j, \alpha_{j+1}]\big)$$

given by taking m to its poles other than ∞_-.

General Finite Gap Set

Moreover, the corresponding half-line Jacobi parameters are almost periodic with frequency module the harmonic measures of the bands.

We'll see parts of where this comes from in the next two lectures. For full details, see Section 5.13 of [SzThm] or the original paper of Christiansen, Simon, Zinchenko [Constr. Approx. **32** (2010), 1–65].

One can also describe this isospectral torus in terms of reflectionless whole-line Jacobi matrices, which I hope to discuss in the final lectures.