Spectral Theory of Orthogonal Polynomials

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Lecture 5: Killip–Simon Theorem on $[-2, 2]$

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References

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[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Killip–Simon Theorem

In 2000, Rowan Killip and I proved the following OPRL analog of Szegő's Theorem.

Theorem. Let $d\mu(x) = f(x)\,dx + d\mu_s$ with Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$. Then

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty$$

if and only if

(i) (Blumental–Weyl) $\sigma_{\text{ess}}(J) = \operatorname{ess\,supp}(d\mu) = [-2, 2]$, i.e., $\operatorname{supp}(d\mu)$ is a set of pure points whose only possible limit points are ± 2: $E_1^- < E_2^- < \dots < -2$; $2 < \dots < E_2^+ < E_1^+$.

(ii) (Lieb–Thirring) $\sum_{\pm,j} (|E_j^\pm| - 2)^{3/2} < \infty$.

(iii) (Quasi-Szegő) $\int (x^2 - 4)^{1/2} \log\big(f(x)\big)\,dx < \infty$.

Killip–Simon Theorem

If J_0 is the Jacobi matrix, $a_n \equiv 1$, $b_n \equiv 0$, the L^2 condition is

$$\mathrm{Tr}\big((J - J_0)^2\big) < \infty$$

Weyl's Theorem says $J - J_0$ compact $\Rightarrow \sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(J_0) = [-2, 2]$.

For Schrödinger operators in 1D (and so on half line), Lieb–Thirring proved (initially for $p > 1/2$, $p = 1/2$ is Weidl and then Hundertmark–Lieb–Thomas)

$$\sum_{E_j, \pm} |E_j^{\pm}|^p \leq C_p \int_0^{\infty} |V(x)|^{p+\frac{1}{2}}$$

Killip–Simon Theorem

Hundertmark–Simon (Killip–Simon for $p = {}^3\!/\!{}_2$)

$$\sum \bigl(|E_j^\pm| - 2\bigr)^{p/2} \le \widetilde{C}_p \sum_{n=0}^{\infty} |a_j - 1|^{p+\frac{1}{2}} + |b_j|^{p+\frac{1}{2}}$$

Quasi-Sezgő because power is $+{}^1\!/\!{}_2$, not $-{}^1\!/\!{}_2$ of Szegő condition.

P_2-Sum Rule

Define F on $\mathbb{R} \setminus [-2,2]$ by

$$F(\beta+\beta^{-1}) = \tfrac{1}{4}\left[\beta^2 - \beta^{-2} - \log(\beta^4)\right];$$

$$F(E) = \frac{1}{2}\int_2^{|E|} (y^2-4)^{\frac{1}{2}}\,dy$$

so $F(E) > 0$ and $F(E) = \frac{2}{3}\big(|E|-2\big)^{\frac{3}{2}} + O\big((|E|-2)^{\frac{5}{2}}\big)$.

Define $G(a) = a^2 - 1 - \log(a^2)$, so

$G(a) > 0$ on $(0,\infty)\setminus\{1\}$; $G(a) = 2(a-1)^2 + O\big((a-1)^3\big)$.

$$Q(\mu) = \tfrac{1}{4\pi}\int_{-2}^{2} \log\Big(\tfrac{\sqrt{4-x^2}}{2\pi f(x)}\Big)\sqrt{4-x^2}\,dx$$

$$= -\tfrac{1}{2} S\big(\mu_0 \mid \mu\big);\ \mu_0 = (a_n \equiv 0, b_n \equiv 0) \text{ measure}$$

P_2-Sum Rule

P_2-Sum Rule:

$$Q(\mu) + \sum F(E_n^\pm) = \sum_{n=1}^{\infty} \left[\tfrac{1}{4}\, b_n^2 + \tfrac{1}{2}\, G(a_n) \right]$$

if $\sigma_{\text{ess}}(\mu) = [-2, 2]$.

RHS $< \infty \Leftrightarrow \sum_{n=1}^{\infty} b_n^2 + (a_n - 1)^2 < \infty$.

LHS $< \infty \Leftrightarrow$ Quasi-Szegő $+ \sum_{n,\pm} \left(|E_n^\pm| - 2 \right)^{\frac{3}{2}} < \infty$.

Thus P_2-sum rule $\Rightarrow$ KS Theorem.

The Method of Step-by-Step Sum Rules

Consider first OPUC. Given μ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$, we define the once stripped measure, μ, by

$$\alpha_j(\mu_1) = \alpha_{j+1}(\mu)$$

i.e., drop α_0 and shift left.

If μ obeys a Szegő condition, so does μ_1 and if $d\mu_1 = f_1 \frac{d\theta}{2\pi} + d\mu_{1,s}$, then

$$1 - |\alpha_0|^2 = \exp\Big(\frac{1}{2\pi} \int \log \Big(\frac{f(\theta)}{f_1(\theta)}\Big) d\theta\Big)$$

The Method of Step-by-Step Sum Rules

This is a little dicey if f_1 or f vanish on sets of positive Lebesgue measure but otherwise makes sense even if μ doesn't obey a Szegő condition.

There is such a "single step" sum rule in general where $\log\left(\frac{f(\theta)}{f_1(\theta)}\right)$ is replaced by a function $G(\theta)$ equal to that log if $f(\theta) \neq 0$ and it can be used to prove Szegő's theorem (see [SzThm], Sections 2.6 and 2.7).

Since the proof uses usc of entropy, it only replaces variational upper bound so for OPUC not so significant.

Still it leads to a higher-order Szegő theorem for OPUC (see [SzThm], Section 2.8).

The Method of Step-by-Step Sum Rules

We'll eventually prove

Theorem (P_2 Step-by-Step Sum Rule). *μ_ℓ has Jacobi parameters $a_j(\mu_\ell) = a_{j+\ell}(\mu)$, $b_j(\mu_\ell) = b_{j+\ell}(\mu)$. Then,*

(a) $\sum_{j,\pm}\big[F(E_j^\pm(\mu)) - F(E_j^\pm(\mu_1))\big]$ *is convergent.*

(b) $\exists\, Q(\mu \mid \mu_1)$ *finite for all μ.*

(c) $Q(\mu) < \infty \Leftrightarrow Q(\mu_1) < \infty$ *and in that case*

$$Q(\mu \mid \mu_1) = Q(\mu) - Q(\mu_1).$$

$$\tfrac{1}{4}b_1^2 + G(a_1) = Q(\mu \mid \mu_1) + \sum_{j,\pm} F(E_j^\pm(\mu)) - F(E_j^\pm(\mu_1))$$

Step-by-Step Sum Rule ⇒ Sum Rule

Step 1. P_2 for finite rank perturbations

If $J - J_0$ has rank n, then $\mu_n = \mu_0$ has $Q(\mu_0) = 0 < \infty$. Thus $Q(\mu) < \infty$. Similarly, the sum of F's is finite.

By iteration, we get P_2 for $\mu_{n-1}, \mu_{n-2}, \dots, \mu$.

Step-by-Step Sum Rule ⇒ Sum Rule

Step 2. Let $J^{(n)}$ have

$a_\ell^{(n)} = a_\ell$, $\ell \le n-1$, $a_\ell^{(n)} = 1$ if $\ell \ge n$,
$b_\ell^{(n)} = b_\ell$, $\ell \le n$, $b_\ell^{(n)} = 0$ if $\ell \ge n+1$,
Let $\mathcal{E}(\mu) = \sum F$'s.

By Step 1,

$$Q(\mu^{(n)}) + \mathcal{E}(\mu^{(n)}) = \frac{1}{4}\sum_{j=1}^{n} b_j^2 + \sum_{j=1}^{n-1} G(a_j)$$

Since $Q = -S$, Q is lsc so $Q(\mu) \le \underline{\lim}\, Q(\mu^{(n)})$.

For j fixed, $E_j^\pm(\mu^{(n)}) \to E_j^\pm(\mu)$, so $\sum_{j,\pm\le m} F\big(E_j^\pm(\mu)\big)$ $\le \underline{\lim}\, \mathcal{E}(\mu^{(n)})$, so $\mathcal{E}(\mu) \le \underline{\lim}\, \mathcal{E}(\mu^{(n)})$.

We have thus proven that $Q(\mu) + \mathcal{E}(\mu) \le \sum_{j=1}^\infty \frac{1}{4} b_j^2$ $+ G(a_j)$.

Step-by-Step Sum Rule ⇒ Sum Rule

Step 3. If $Q(\mu)<\infty$, $\mathcal{E}(\mu)<\infty$, by step-by-step, $Q(\mu_n)<\infty$, $\mathcal{E}(\mu)<\infty$ and

$$Q(\mu)+\mathcal{E}(\mu)=\sum_{j=1}^{n-1}\left[\tfrac{1}{4}\,b_j^2+G(a_j)\right]+Q(\mu_n)+\mathcal{E}(\mu_n)$$

$$\geq \sum_{j=1}^{n-1}\tfrac{1}{4}\,b_j^2+G(a_j)$$

Taking $n\to\infty$, $Q(\mu)+\mathcal{E}(\mu)\geq\sum_{j=1}^{\infty}\frac{1}{4}\,b_j^2+G(a_j)$

If $Q=\infty$ or $\mathcal{E}=\infty$, this inequality is trivial.

QED!

m-Functions

One defines $m_\mu(z) = \int \frac{d\mu(x)}{x-z}$ for $z \notin \mathrm{supp}(\mu) = \sigma(J)$.

Of course, m is analytic on $\mathbb{C} \setminus \sigma(J)$ and meromorphic at isolated pure points of μ.

Moreover, since J is multiplication by x in $L^2(\mathbb{R}, d\mu)$, isolated eigenvalues of J are exactly the poles of m_μ.

We'll see soon that the poles of m_{μ_1}, the once-stripped m are precisely the zeros of m_μ.

m-Functions

If $(\alpha, \beta) \subset \mathbb{R} \setminus \sigma(J)$, $\frac{dm(y)}{dy} = \int \frac{d\mu(x)}{(x-y)^2} > 0$ so

zeros and poles of m interlace. Since $m \to 0$ at $\pm\infty$, last "pole or zero" is a pole.

Thus, $E_1^+(\mu) > E_1^+(\mu_1) > E_2^+(\mu) > E_2^+(\mu_1) \dots$

$\Rightarrow$ terms in $F\big(E(\mu)\big) - F\big(E(\mu_1)\big)$ are all positive

and as alternating sum, the sum converges.

Also for Lebesgue a.e. x, $f(x) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im} m(x + i\varepsilon)$

Second Kind Polynomials

$$q_n(x) = \int \frac{p_n(x) - p_n(y)}{x - y}\, d\mu(y); \quad q_0 = 0,\, q_{-1} = -1$$

Since $p_1(x) = a_1^{-1}(x - b_1)$, we have $q_1 = a_1^{-1}$.

Using recursion relation for p's, see q obeys same relations.

Indeed,

$$q_n(x) = a_1^{-1} p_{n-1}\big(x; \{a_{\ell+1}, b_{\ell+1}\}_{\ell=0}^{\infty}\big)$$

are "essentially" the p's for $d\mu_1$.

Weyl Solution

For $z \notin \sigma(J)$, define the Weyl solution

$g_n(z) \equiv m(z)\, p_n(z) + q_n(z)$

which is a solution of difference equation. Thus,

$$g_n(z) = p_n(z) \int \frac{d\mu(x)}{x-z} - p_n(z) \int \frac{d\mu(x)}{x-z} + \int \frac{p_n(x)}{x-z}\, d\mu(x)$$

$$= \langle p_n, (\cdot - z)^{-1} \rangle$$

Since $(\cdot - z)^{-1} \in L^2(\mathbb{R}, d\mu)$, we see

$$\sum_{n=0}^{\infty} |g_n(z)|^2 < \infty \quad \left(= \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} \text{ if } \operatorname{Im} z \neq 0 \right)$$

If $\inf_n a_n > 0$, the Weyl solution is the unique L^2 solution (up to a constant) by constancy of the Wronskian.

Weyl Solution

Clearly $m(z) = -\frac{g_0(z)}{a_0\, g_{-1}(z)}$

since $q_0 = 0$, $p_0 = 1$, $q_{-1} = -1$, $p_{-1} = 0$, $a_0 = 1$.

By uniqueness of L^2 solutions up to a constant

$g_n(z; d\mu_1) = c(z)\, g_{n+1}(z; d\mu), n \geq -1.$

Thus, $m(z; d\mu_1) = \frac{-g_1(z)}{a_1\, g_0(z)}$.

In $m(z) = -g_0/a_0\, g_{-1}$, we put $a_0 = 1$, but it works for any value of a_0 which is why we put in the a_1.

Since $a_1\, g_1 + (b_1 - z)g_0 + a_0\, g_{-1} = 0$, we see that

$-a_1^2\, m_1 + (b_1 - z) - m(z)^{-1} = 0.$

Weyl Solution

Thus, $m(z) = (b_1 - z - a_1^2\, m_1(z))^{-1}$, called the coefficient stripping relation.

In particular, poles of m_1 are exactly the zeros of m as we claimed.

Iterating gives Markov continued fraction expansion for m !

In particular taking $z = x + i\varepsilon$, $\varepsilon \downarrow 0$ using $\operatorname{Im}(w^{-1}) = -\operatorname{Im} w/|w|^2$,

$$\varepsilon + a_1^2 \operatorname{Im} m_1 = \operatorname{Im} m/|m|^2 \Rightarrow f/f_1 = |a_1 m|^2$$

Meromorphic Herglotz Functions on $\mathbb{D}$

Let M be meromotphic on $\mathbb{D}$ with $\pm \operatorname{Im} M > 0$ if $\pm \operatorname{Im} z > 0$. Then poles and zeros (i.e., on $(-1,1)$) interlace. By controlling the ratio of Blaschke products as zeros move, one proves that

Theorem. *If* $\{z_j\}_{j=1}^\infty$, $\{p_j\}_{j=1}^\infty \subset (-1,1)$ *with* $|z_j| \to 1$ *as* $j \to \infty$ *and* $\sum_{j=1}^\infty |z_j - p_j| < \infty$ *(automatic if interlaced), then*

$$\prod_{j=1}^N \frac{b_{z_j}(z)}{b_{p_j}(z)} \to B(z)$$

as meromorphic functions on $\mathbb{D}$ *"uniformly" (as functions to Riemann sphere).*

B converges in UHP uniformly on compacts, so $|B(e^{i\theta})| = 1$.

As usual $b_{w=0}(z) = z$, $b_{w\neq 0}(z) = -\frac{|w|}{w}\,\frac{z-w}{1-\bar{w}z}$.

Meromorphic Herglotz Functions on $\mathbb{D}$

Let B_∞ be Blaschke product of zeros and poles for M, a meromorphic Herglotz function on $\mathbb{D}$. One proves in UHP, $|\arg B_\infty(z)| \leq 2\pi$ (starting from $\arg B_\infty(x) = 0$ for $B_\infty(x) > 0$ on $\mathbb{R}$) so $\arg(M/B_\infty)$ is bounded, so by M. Riesz Theorem,

$$\log\left(M/B_\infty\right) \subset \cap_{p<\infty} H^p$$

We get

Theorem. *If M is a meromorphic Herglotz function on $\mathbb{D}$, B_∞ meromorphic on $\mathbb{D}$, poles only at poles of M. Then for a.e. θ, $\lim_{r\uparrow 1} M(re^{i\theta}) \equiv M(e^{i\theta})$ exists with*

$$\int \Big[\log|M(e^{i\theta})|\Big]^p \frac{d\theta}{2\pi} < \infty$$

for all $p \in [1,\infty)$

with

$$f(z) = \sigma B_\infty(z) \exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|M(e^{i\theta})| \frac{d\theta}{2\pi}\biggr)$$

where $\sigma = \pm 1$.

$\sigma = \operatorname{sgn}\bigl(f(0)\bigr)$ *if* $f(0) \neq 0$, $\sigma = 1$ *if* $f(0) = 0$, $\sigma = -1$ *if* $f(0) = \infty$.

Step-by-Step Case Sum Rules

(named after Ken Case) We now apply this to

$$M(z) = m(z + z^{-1})$$

looking at $\log\left(\frac{a_1 M(z)}{z}\right)$. $z = 0$ corresponds to $x = \infty$, so there are Taylor coefficients expressible in terms of continued function expansion.The leading terms are

$$\log \frac{a_1 M(z)}{z} = \log a_1 + b_1 z + (\tfrac{1}{2} b_1^2 + a_1^2 - 1) z^2 + O(z^3)$$

Using

$$\log\left(1 - \frac{\beta}{z + z^{-1}}\right) = \sum_{n=1}^{\infty} \tfrac{2}{n} \left[T_n(0) - T_n(\tfrac{1}{2}\beta)\right] z^n$$

one can obtain “explicit” formulas for the Taylor coefficients.

Step-by-Step Case Sum Rules

One also expands the log of Blaschke terms, using

$\log b_w(z) = \log|w| + \sum_{n=1}^\infty \frac{z^n}{n}(w^n - w^{-n})$

and $\frac{e^{i\theta}+z}{e^{i\theta}-z} = 1 + 2\sum_{n=1}^\infty z^n e^{-in\theta}$.

One gets the C_0 step-by-step rule

$$-\log(a_1) = Z(J \mid J_1) + \sum_{j,\pm}\left[\log(|p_j|) - \log(|z_j|)\right]$$

$$Z(J \mid J_1) = \frac{1}{4\pi}\int_0^{2\pi} \log\left(\frac{\operatorname{Im} M_1(e^{i\theta})}{\operatorname{Im} M(e^{i\theta})}\right) d\theta$$

if $\operatorname{Im} M(e^{i\theta}) \neq 0$, otherwise it's really $|a_1\, M(e^{i\theta})|^2$.

Step-by-Step Case Sum Rules

For $n \geq 1$, C_n sum rules,

$\mathcal{P}_n(a\text{'s},\, b\text{'s}) = S_n + \mathcal{E}_n$

$\mathcal{P}_n$ is, in general, complicated but

$\mathcal{P}_2 = \frac{1}{2}\, b_1^2 + a_1^2 - 1, \quad \mathcal{P}_1 = b_1$

$\mathcal{E}_n = \sum_{j,\pm 1} \frac{z_j^n - p_j^n - (z_j^{-n} - p_j^{-n})}{n}$

$S_n = -\frac{1}{2\pi} \int_0^{2\pi} \log\left(\frac{\operatorname{Im} M_1(e^{i\theta})}{\operatorname{Im} M(e^{i\theta})}\right) \cos(n\theta)\, d\theta$

where we use $\operatorname{Im} M(e^{i\theta}) = -\operatorname{Im} M(e^{i\theta})$ so ratio is even to replace $e^{in\theta}$ by $\cos(n\theta)$.

The End of the Story

$\mathcal{P}_2$ is $C_0 + \frac{1}{2}C_2$. A miracle takes place!

$\frac{1}{4\pi} - \frac{1}{4\pi}\cos(2\theta) = \frac{1}{2\pi}\sin^2\theta$, so the entropies terms combine to

$$Q(J \mid J_1) = \frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{\operatorname{Im} M_1}{\operatorname{Im} M}\right)\sin^2\theta\, d\theta$$

$$-\log(a_1) + \tfrac{1}{2}\left(\tfrac{1}{2}b_1^2 + a_1^2 - 1\right) = \tfrac{1}{4}\, b_1^2 + \tfrac{1}{2}\, G(a_1)$$

with $G(a) > 0$ on $(0,\infty)\setminus\{1\}$.

The Blaschke terms also combine to something positive.

Everything works because of the positivity. So far, there is no understanding why they are positive other than as a fortuitous result of calculation!