Spectral Theory of Orthogonal Polynomials

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Lectures 3 & 4: Three Kinds of Polynomial Asymptotics, I, II

- Lecture 2: Szegö Theorem for OPUC
- Lecture 3: Three Kinds of Polynomials Asymptotics, I
- Lecture 4: Three Kinds of Polynomial Asymptotics, II
- Lecture 5: Killip–Simon Theorem on $[-2,2]$

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Asymptotics of Chebyshev of Second Kind

Since

$$\sin(n \pm 1)\theta = \sin n\theta \, \cos\theta \pm \cos n\theta \, \sin\theta$$

we have that

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta(\sin n\theta)$$

If $f_n(\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$, then $f_{-1} = 0$, $f_0 = 1$, and $f_{n+1} + f_{n-1} = (2\cos\theta) f_n$.

Thus, by induction, $f_n(\theta)$ is a polynomial in $2\cos\theta$ of degree n, i.e.,

$$f_n(\theta) = p_n(2\cos\theta)$$

where

$$p_{n+1}(x) + p_{n-1}(x) = x p_n(x); \quad p_{-1} = 0, \, p_0 = 1$$

Asymptotics of Chebyshev of Second Kind

Thus, $\{p_n(x)\}_{n=0}^{\infty}$ are the orthonormal OPs with Jacobi parameters, $b_n \equiv 0$, $a_n \equiv 1$.

$x = 2\cos\theta$ (leads to quadratic equation for $e^{i\theta}$) so

$$e^{\pm i\theta} = \frac{x}{2} \pm \sqrt{1 - \left(\frac{x}{2}\right)^2}$$

WARNING: I am very bad at calculations. Factors of $2, \pi$, etc., could be wrong.

Since $\sin(k\theta)$ are orthogonal for $\frac{d\theta}{2\pi}$, $f_n(\theta)$ are orthogonal for $\sin^2\theta \frac{d\theta}{2\pi}$ (for normalization on $[0, 2\pi]$).

Asymptotics of Chebyshev of Second Kind

But $\theta \mapsto x = 2\cos\theta$ is 2 to 1 from $[0, 2\pi]$ to $[-2, 2]$, so we want to look at $2\sin^2\theta \frac{d\theta}{2\pi}$ on $[0, \pi]$.

$x = 2\cos\theta \Rightarrow dx = 2\sin\theta d\theta$, so the measure is $\sin\theta dx = \sqrt{1 - \left(\frac{x}{2}\right)^2} dx$, i.e.,

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, dx$$

is the orthogonality measure for this problem.

Asymptotics of Chebyshev of Second Kind

If $x \notin [-2,2]$ ($x \in \mathbb{C}$), $e^{\pm i\theta}$ have different rates of growth so one dominates for $\sin(n+1)\theta / \sin\theta$ for n large, i.e.,

$$|p_n(x)| \Bigg/ \left| \frac{x}{2} + \sqrt{1 - \left(\frac{x}{2}\right)} \right|^n \to 1$$

as $n \to \infty$. $x \notin [-2,2]$ is critical to avoid oscillation.

There is a branch of $\sqrt{\quad}$ so $|\cdots| > 1$ on $\mathbb{C} \setminus [-2,2]$.

One question we'll answer is where $\frac{x}{2} + \sqrt{1 - \left(\frac{x}{2}\right)^2}$ comes from.

Three Kinds of Asymptotics

What does it mean to say that a sequence, $y_n \sim a^n$ for n large?

Root asymptotics: $|y_n|^{1/n} \to |a|$.

Ratio asymptotics: $\frac{y_{n+1}}{y_n} \to a$.

Szegő asymptotics: $y_n / A a^n \to 1$ for some A.

Three Kinds of Asymptotics

A second theme in this pair of lectures will be to explore when these conditions hold for OPUC/OPRL close to the "free" case ($\alpha_n \equiv 0$ for OPUC; $a_n \equiv 1, b_n \equiv 0$ for OPRL).

We'll look at this asymptotics away from $\mathrm{supp}(d\mu)$ because on $\mathrm{supp}(d\mu)$, the asymptotics are typically unusual (decay rather than growth for isolated points in $\mathrm{supp}(d\mu)$; oscillation on the a.c. part of $d\mu$.)

That said, asymptotic behavior on the spectrum can have important consequences as we'll illustrate with the theory of L^1 perturbations.

OPUC Transfer Matrices

We begin by looking at all solutions of the difference equations that describe recursion. In some sense, they are both second order, so there is a 2×2 "update" matrix that takes data at $n = 0$ to data at $n = m$.

For OPUC, we saw that $A(z;\alpha_n)\binom{\varphi_n}{\varphi_n^*} = \binom{\varphi_{n+1}}{\varphi_{n+1}^*}$

$$A(z;\alpha) = \rho^{-1}\begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix}$$

Notice that $\det A(z;\alpha) = z$, so for $z \neq 0$, $z \in \mathbb{C}$, we have A invertible and for $z \in \partial\mathbb{D}$,

$$\|A^{-1}\| = \|A\|$$

OPUC Transfer Matrices

Define the transfer matrix by

$$T_n(z;\alpha_{n-1},\dots,\alpha_0) = A(z;\alpha_{n-1})\,A(z;\alpha_{n-2})\cdots A(z;\alpha_0)$$

Thus,

$$\begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} = T_n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The second kind of polynomials are defined by

$$\begin{pmatrix} \psi_n \\ -\psi_n^* \end{pmatrix} = T_n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A little thought using

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A(z;\alpha) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A(z;-\alpha)$$

shows that

$$\psi_n\big(z;\{\alpha_j\}_{j=0}^{n-1}\big) = \varphi_n\big(z;\{-\alpha_j\}_{j=0}^{n-1}\big)$$

OPUC L^1 Perturbation

As a simple application of transfer matrices for OPUC, we prove

Theorem. *If*

$$\sum_{j=0}^{\infty} |\alpha_j| < \infty$$

then $d\mu = w(\theta)\frac{d\theta}{2\pi}$ *with* $\inf w > 0$, $\operatorname{supp} w < \infty$ *(so* $d\mu_s = 0$*).*

Remarks. 1. Our proof can be slightly extended to show w is continuous.

2. A much stronger result is known (Baxter's Theorem): $\sum_{j=0}^{\infty}|\alpha_j(d\mu)| < \infty \Leftrightarrow \sum_{j=0}^{\infty}|c_j(d\mu)| < \infty + (d\mu = w(\theta)\frac{d\theta}{2\pi}$, w continuous with $\inf w > 0$.)

OPUC L^1 Perturbation

Notice that for $|z|=1$, we have that (Euclidean norm on $\mathbb{C}^2$)

$$\|A(z;\alpha)\| \leq 1+|\alpha| \leq e^{|\alpha|}$$

Thus, $\|T_n(z;\alpha_0,\cdots\alpha_{n-1})\| \leq e^{\sum\limits_0^{n-1}|\alpha_j|}$

so $\sup_{|z|=1,n}|\varphi_n(t)| \leq e^{\sum\limits_0^{\infty}|\alpha_j|}$

but $\|A^{-1}\| = \|A\|$ for $|z|=1$ and $|\varphi| = |\varphi^*|$

implies $\inf_{|z|=1,n}|\varphi_n(t)| \geq e^{-\sum\limits_0^{\infty}|\alpha_j|}$

Thus, by Bernstein–Szegő, we get the desired result.

OPRL Transfer Matrix

Consider the difference equation

$$u_{n+1} = a_n^{-1}\big((z - b_n)u_n - a_{n-1}u_{n-1}\big)$$

$u_n = p_{n-1}(z)$ solves this equation with $u_0 = 0$, $u_1 = 1$.

The difference equation can be rewritten (we take $a_0 = 1$)

$$\begin{pmatrix} u_{n+1} \\ a_n u_n \end{pmatrix} = A(z; a_n, b_n) \begin{pmatrix} u_n \\ a_{n-1}\, u_{n-1} \end{pmatrix};$$

$$A(z; a, b) = \frac{1}{a} \begin{pmatrix} z - b & -1 \\ a^2 & 0 \end{pmatrix}$$

OPUC L^1 Perturbation

The reason for the funny a_n in the lower component (a suggestion of Killip) is that it makes

$$\det A = 1$$

This implies if u, v are two solutions (same z) that (courtesy of Wronkian) $a_n(u_{n+1}v_n - u_n v_{n+1}) =$ constant.

As for OPUC, we define

$T_n(z; \{a_j, b_j\}_{j=1}^n) = A(z; a_n, b_n) \cdots A(z : a_1, b_1)$ so

$$T_n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_n(z) \\ a_n p_{n-1}(z) \end{pmatrix}$$

OPRL L^1 Perturbation

In the free Jacobi matrix case,

$$A_0(z) = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$$

Since $\|A_0(z)\binom{1}{0}\| = \|\binom{z}{1}\| = 1 + |z|^2$, except for $z = 0$, $A_0(z)$ is not a contraction in the Euclidean norm. Since (as we'll see) $\sup_n \|A_0(z)^n\|$ is bounded for $z \in (-2, 2)$, this isn't a problem for A_0 but it makes perturbations tricky.

We'll overcome this by changing norm. In essence, the plane wave solutions will be a basis, so this is essentially a variation of parameters argument.

OPRL L^1 Perturbation

We are heading towards a proof of

Theorem. *Let* $\{a_n, b_n\}_{n=1}^\infty \subset \big[(0,\infty)\times\mathbb{R}\big]^\infty$ *obey*

$$\sum_{n=1}^\infty |a_n - 1| + |b_n| < \infty$$

Then, for any $\varepsilon > 0$*, there is* $C_\varepsilon > 0$ *so that for all* n *and all* $x \in [-2+\varepsilon, 2-\varepsilon]$*, we have*

$$C_\varepsilon \le |p_n(x)|^2 + |p_{n-1}(x)|^2 \le C_\varepsilon^{-1}$$

In particular (since $0 < \inf a_n < \sup a_n < \infty$*),* J *has purely a.c. spectrum in* $(-2,2)$*.*

OPRL L^1 Perturbation

Since $\det A_0(2\cos\theta) = 1$, $\mathrm{Tr}\big(A_0(2\cos\theta)\big) = 2\cos\theta$, the eigenvalues of $A_0(2\cos\theta)$ are $\pm e^{i\theta}$. Thus, for $x \in (-2, 2)$, there is $U(x)$ so

$$U(x)\, A_0(x)\, U(x)^{-1} = \begin{pmatrix} e^{i\theta(x)} & 0 \\ 0 & e^{-i\theta(x)} \end{pmatrix}$$

We define

$$\|B\|_x = \|U(x)BU(x)^{-1}\|$$

where $\|\cdot\|$ without an x is Euclidean norm. $\|\cdot\|_x$ is a Banach algebra norm on $\mathrm{Hom}(\mathbb{C}^2)$, since

$$U(x)BCU(x)^{-1} = \big[U(x)BU(x)^{-1}\big]\big[U(x)CU(x)^{-1}\big]$$

OPRL L^1 Perturbation

$U(x)$ is singular at $x = \pm 2$ but on $(-2,2)$ it can be chosen real analytic (and, in particular, so $U(x)$ and $U(x)^{-1}$ are bounded on each $[-2+\varepsilon, 2-\varepsilon]$).

Thus, for each interval, there is $D_\varepsilon > 0$ so for all x in the interval and B

$$D_\varepsilon \|B\| \leq \|B\|_x \leq D_\varepsilon^{-1} \|B\|$$

The point, of course, is that $\|A_0(x)\|_x = 1$, so

$$\|a_n A_n(x; a_n, b_n)\|_x \leq 1 + E_x \big[\|a_n - 1\| + \|b_n\| \big]$$

OPRL L^1 Perturbation

Since $\delta \leq a_n \leq \delta^{-1}$ and $\sum_n |a_n - 1| < \infty$, $\prod_{j=1}^n a_j$ and its inverse converge and are uniformly bounded.

We conclude $\|T_n\|_x$ and $\|T_n^{-1}\|_x$ and so $\|T_n\|$ and $\|T_n^{-1}\|$ are uniformly bounded on $[-2 + \varepsilon, 2 + \varepsilon]$ which yields the claimed estimates.

Szegő Asymptotics for OPUC

For OPUC, the condition for $d\mu = f(\theta)\frac{d\theta}{2\pi} + d\mu_s$

$$\int \log f(\theta)\frac{d\theta}{2\pi} > -\infty$$

is called the Szegő condition. When it holds, we define the Szegő function, $D(z)$, on $\mathbb{D}$ by

$$D(z) = \exp\left(\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\bigl(f(\theta)\bigr)\frac{d\theta}{4\pi}\right)$$

Lemma. *If the Szegő condition holds, $D \in H^2(\mathbb{D})$, indeed,*

$$\sup_{0 \le r < 1} \int |D(re^{i\theta})|^2 \frac{d\theta}{2\pi} \le 1$$

and, with $D(e^{i\theta}) \equiv \lim_{r \uparrow 1} D(re^{i\theta})$,

$$|D(e^{i\theta})|^2 = f(\theta)$$

Szegő Asymptotics for OPUC

Proof. Let $f_\varepsilon(\theta) = \min\big(f(\theta), \varepsilon^{-1}\big)$. Then $\log\big(f_\varepsilon(\theta)\big)$ is bounded above by $\log(\varepsilon^{-1})$, so

$$\operatorname{Re}\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\big(f_\varepsilon(\theta)\big) \frac{d\theta}{4\pi}\biggr) \le \frac{1}{2}\log(\varepsilon^{-1})$$

so $|D_\varepsilon| \le \varepsilon^{-1/2}$. Thus, D_ε lies in H^∞ and has boundary values

$$|D_\varepsilon(e^{i\theta})|^2 = f_\varepsilon(\theta)$$

Therefore, $D_\varepsilon \in H^2$ and

$$\sup_{0\le r<1} \int |D_\varepsilon(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \int |D_\varepsilon(e^{i\theta})|^2 \frac{d\theta}{2\pi} \le 1$$

Taking $\varepsilon \downarrow 0$, we see that $D \in H^2$ and the rest follows.

Szegő Asymptotics for OPUC

We have the following beautiful calculation of Szegő:

$$\int |\varphi_n^*(e^{i\theta})\, D(e^{i\theta}) - 1|^2 \frac{d\theta}{2\pi} + \int |\varphi_n^*(e^{i\theta})|^2 d\mu_s = 2\left(1 - \prod_{j=n}^\infty \rho_j\right)$$

For

$$\text{LHS} = \int \frac{d\theta}{2\pi} + \int |\varphi_n^*(e^{i\theta})|^2 d\mu - 2\,\mathrm{Re} \int D(e^{i\theta})\varphi_n^*(e^{i\theta}) \frac{d\theta}{2\pi}$$

$$= 2 - 2\,\mathrm{Re}\big(D(0)\varphi_n^*(0)\big)$$

$$= 2\left[1 - \prod_{j=0}^\infty \rho_j \left(\prod_{j=0}^{n-1} \rho_j^{-1}\right)\right]$$

Szegő Asymptotics for OPUC

Since RHS $\to 0$ as $n \to \infty$ (if the product converges, i.e., if the Szegő condition holds), each term goes to zero.

Thus $\int |\varphi_n^*(e^{i\theta})|^2 d\mu_s \to 0$ and $\varphi_n^* \, D \to 1$ in $L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$.

Since the Poisson kernel $P_z(e^{i\theta})$ is L^2 uniformly for $|z| \leq r < 1$, $\varphi_n^*(z) \, D(z) \to 1$ uniformly on $|z| \leq r < 1$.

Thus, uniformly in $|z| \geq r^{-1} > 1$,

$$z^{-n} \varphi_n(z) \to \left[\overline{D\left(\frac{1}{\bar{z}}\right)} \right]^{-1}$$

which is Szegő asymptotics for φ_n.

The Szegő Mapping

We now turn to OPRL with μ supported on $[-2, 2]$. Since we'll later consider a related result which generalizes this, we'll only sketch or, even hand wave, some details.

The map

$$z \mapsto x = z + z^{-1}$$

(called the Joukowski map) is a 2 to 1 map of $\partial\mathbb{D}$ to $[-2, 2]$ that takes $e^{i\theta}$ to $2\cos\theta$ in the limit.

The Szegő Mapping

$Q(e^{i\theta}) = 2\cos\theta$ induces a map of $C\big([-2,2]\big)$ to $C(\partial\mathbb{D})$ by $\big(Qf\big)(e^{i\theta}) = f\big(Q(e^{i\theta})\big)$. It is onto the even functions, i.e., $g(e^{-i\theta}) = g(e^{i\theta})$. By duality, it defines a dual map Sz:

Even measures on $\partial\mathbb{D}$ to some probability measures on $[-2,2]$ by $d\rho = \mathrm{Sz}(d\mu)$

$$\int f\big(\arccos(\frac{x}{2})\big)\, d\rho(x) = \int f(\theta)\, d\mu(\theta)$$

The Szegő Mapping

Let P_n be the monic OP's for $d\rho = \mathrm{Sz}(d\mu)$ and Φ_n for μ. Then

$$P_n(z + \frac{1}{z}) = \left[1 - \alpha_{2n-1}(d\mu)\right]^{-1} z^{-n} \left[\Phi_{2n}(z) + \Phi_{2n}^*(z)\right]$$

This can be proven by noting first that the right side is a Laurent polynomial of z, even under $z \to \frac{1}{z}$ and every such Laurent polynomial has the form $Q_n(z + \frac{1}{z})$.

The Szegő Mapping

By an easy computation $\int(\text{RHS for } n)\,(\text{RHS for } \ell)\,d\mu = 0$ if $n \neq \ell$, so the Q_n's are OP and by the leading term, it is monic.

By computing $\langle \Phi_{2n}, \Phi_{2n}^* \rangle = -\alpha_{2n-1} \|\Phi_{2n}\|^2$, one finds

$$\|P_n\|_{L^2(d\rho)}^2 = 2(1 - \alpha_{2n-1})^{-1} \|\Phi_{2n}\|_{L^2(d\mu)}^2$$

This implies that

$$(a_1 \cdots a_n)^2 = 2(1 + \alpha_{2n-1}) \prod_{j=0}^{2n-2} (1 - \alpha_j^2)$$

The Szegő Mapping

One also finds (Section 13.1 and 13.2 of [OPUC2] have two different proofs)—known as Geronimus relations

$$a_{n+1}^2 = (1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1})$$

$$b_{n+1} = (1 - \alpha_{2n-1})\alpha_{2n} - (1 + \alpha_{2n-1})\alpha_{2n-2}$$

Szegő Asymptotics for $[-2, 2]$

From $a_n^2 \cdots a_1^2 = 2(1 + \alpha_{2n-1}) \prod_{j=0}^{2n-1} (1 - \alpha_j^2)$, one sees

$$\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty \Leftrightarrow \lim\ \sup\ a_1 \cdots a_n > 0$$

This leads to

Shohat–Nevai Theorem. *Let* $d\mu = f(x)\, dx + d\mu_s$ *be supported on* $[-2, 2]$. *Then*

$$\lim\ \sup\ a_1 \cdots a_n > 0 \Leftrightarrow \int_{-2}^{2} (4 - x^2)^{-1/2} \log\bigl(f(x)\bigr)\, dx > -\infty$$

If that holds, then

$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty$, $\lim a_1 \cdots a_N$,

$\lim \sum_{n=1}^{N} (a_n - 1)$ *and* $\lim \sum_{n=1}^{N} b_n$ *all exist.*

Szegő Asymptotics for $[-2,2]$

It is critical that we require that support$(d\mu) \subset [-2,2]$, i.e., no eigenvalues outside $[-2,2]$—unnatural from perturbation theory point of view.

$\int_{-2}^{2}(4-x^2)^{-1/2}\log\bigl(f(x)\bigr)\,dx > -\infty$ is called *the Szegő condition*.

$x = 2\cos\theta \Rightarrow dx = 2\sin\theta d\theta \Rightarrow d\theta = \frac{dx}{2\sin(\theta)}$
$\Rightarrow d\theta = (4-x^2)^{-1/2}\,dx.$

The other relations follow from Geronimus relations.

Szegő Asymptotics for $[-2, 2]$

Recall that

$$P_n(z + \frac{1}{z}) = \left[1 - \alpha_{2n-1}(d\mu)\right]^{-1} z^{-n} \left[\Phi_{2n}(z) + \Phi_{2n}^*(z)\right]$$

and for $|z| > 1$,

$$z^{-2n}\Phi_{2n}(z) \to D(0)/\overline{D\left(\frac{1}{\bar{z}}\right)}$$

By the maximum principle $(1 + \varepsilon)^{-2n}\Phi_{2n}(z) \to 0$ for $|z| > 1$, so $z^{-2n}\Phi_{2n}^*(z) \to 0$.

Thus, we obtain

Szegő Asymptotics for $[-2, 2]$

Theorem (Szegő asymptotics for $[-2, 2]$, with no bound states). *If the Szegő condition holds, then, for* $|z| > 1$

$$z^{-n} P_n\left(z + \frac{1}{z}\right) \to G(z) \equiv \left[1 - \alpha_{2n-1}(d\mu)\right]^{-1} D(0)/\overline{D\left(\frac{1}{z}\right)}$$

Equivalently, for $x \in \mathbb{C} \setminus [-2, 2]$

$$\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right) - 1}\right)^{-n} P_n(x) \to \widetilde{G}(x)$$

The Density of Zeros

I now say a little about root and ratio asymptotics. In the final lectures, I hope to return to this subject.

As a warm-up for root asymptotics, let J_N be the $N \times N$ truncated Jacobi matrix (with $b_1, \dots, b_n$ along the diagonal). Let $D_n(z) = \det(z - J_N)$. Then, expanding in minors:

$$D_N = -a_{N-1}^2 D_{N-2} + (z - b_N) D_{N-1}; \quad D_0 = 1,\ D_{-1} = 0$$

Thus $D_N(z) = P_N(z)$.

which implies zero of P_N = eigenvalues of J_N are real and simple.

The Density of Zeros

For each N, let $x_1^{(N)} < \cdots < x_N^{(N)}$ be the zeros. By the variational principle, $x_j^{(N)} < x_j^{(N+1)} < x_{j+1}^{(N+1)}$, i.e., zero interlace. Let

$$\nu^{(N)} = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j^{(N)}}$$

If

$$\nu = \text{w-lim}\, \nu^{(N)}$$

exists, we say ν is the density of zeros, aka, density of states.

The Density of Zeros

ν is boundary condition independent, e.g., if

$$J_N^{\text{per}} = \begin{pmatrix} b_1 & \dots & a_N e^{i\theta} \\ \vdots & \ddots & \vdots \\ a_N e^{-i\theta} & \dots & b_N \end{pmatrix}$$

$$\text{w-lim}\, \nu_{\text{per}}^{(N)} = \text{w-lim}\, \nu^{(N)}$$

For

$$\int x^\ell \, d\nu(x) = \lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}(J_n^\ell)$$

and $|\operatorname{Tr}(J_N^\ell) - \operatorname{Tr}\big((J_N^{\text{per}})^\ell\big)|$ is bounded.

Thouless Formula

The DOS is intimately connected to root asymptotics because

$$p_n(z) = (a_1 \cdots a_n)^{-1} \prod_{j=1}^{N} (z - x_j^{(n)})$$

so

$$\frac{1}{n} \log |p_n(z)| = -\frac{1}{n} \log (a_1 \cdots a_n) + \int \log |z-x| \, d\nu^{(N)}(x)$$

Theorem (Thouless Formula). If DOS exists and

$$\lim (a_1 \cdots a_n)^{1/n} = c(d\mu)$$

exists, then for $z \in \mathbb{C} \setminus \mathbb{R}$, ($\Phi_\mu(z) = \int \log |z-x|^{-1} \, d\mu(x)$ is the potential of μ)

$$\lim \frac{1}{n} \log |p_n(z)| = -\log c(d\mu) + \int \log |z-x| \, d\nu(x)$$

Connection to Potential Theory

Given any compact set, $\mathfrak{e}$, we say $\mathfrak{e}$ has zero capacity if

$$\mathcal{E}(\mu) = \int d\mu(x)\, d\mu(y)\, \log|x-y|^{-1}$$

is infinite for all $\mu \in M_{+,1}(\mathfrak{e})$.

(Note: the integral is either $+\infty$ or finite.)

If $\mathfrak{e}$ does not have zero capacity, we define $C(\mathfrak{e})$ by

$$C(\mathfrak{e}) = \exp\big(-\inf_{\mu\in M_{+,1}(\mathfrak{e})} \mathcal{E}(\mu)\big)$$

Connection to Potential Theory

It is a fundamental theorem that if $C(\mathfrak{e}) > 0$, there is a unique probability measure, $\rho_{\mathfrak{e}}$, called the *equilibrium measure* or the *harmonic measure* for $\mathfrak{e}$ with $\mathcal{E}(\rho_{\mathfrak{e}}) = \inf \mathcal{E}(\mu)$.

$T_{n,\mathfrak{e}}$, the Chebyschev polynomial for $\mathfrak{e}$, is the (it turns out unique) monic polynomial of degree n with

$$\|T_{n,\mathfrak{e}}\|_{\infty,\mathfrak{e}} = \inf_{P \text{ monic}} \|P\|_{\infty,\mathfrak{e}}; \quad \|f\|_{\infty,\mathfrak{e}} = \sup_{x\in\mathfrak{e}} |f(x)|$$

Theorem (Faber–Fekete-Szegő).

$$\|T_n\|_{\infty,\mathfrak{e}}^{1/n} \geq C(\mathfrak{e}) \text{ and } \lim_{n\to\infty} \|T_n\|_{\infty,\mathfrak{e}}^{1/n} = C(\mathfrak{e})$$

Regular Measures

Since $\|T_n\|_{L^2(d\mu)} \leq \|T_n\|_{\infty,\mathfrak{e}}$, if

$$\mathfrak{e} = \operatorname{supp}(\mu)$$

and $\|P_n\|_{L^2(d\mu)} \leq \|T_n\|_{L^2(d\mu)}$ (by variational principle)

$\limsup (a_1 \cdots a_n)^{1/n} \leq C(\mathfrak{e})$.

We call μ regular (with $\operatorname{supp}(\mu) = \mathfrak{e} \subset \mathbb{R}$) if $\lim_{n\to\infty} (a_1 \cdots a_n)^{1/n} = C(\mathfrak{e})$.

Pioneers are Ulmann (for $\mathfrak{e} = [0, 1]$) and Stahl–Totik ($\mathfrak{e} \in \mathbb{C}$).

See also Simon, *Inv. Prob. Imaging* **1** (2007), 189–215.

Regular Measures

If μ is regular, the DOS exists and equals the equilibrium measure for $\mathfrak{e}$.

Thus, for $z \in \mathbb{C} \setminus \mathbb{R}$, $\lim_{n\to\infty} |p_n(z)|^{1/n} = e^{G_{\mathfrak{e}}(z)}$.

$G_{\mathfrak{e}}(z) = \log \left(C(\mathfrak{e})\right)^{-1} - \Phi_{\rho_{\mathfrak{e}}}(z)$

This is the potential theorists' Green's Function, the unique function subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus \mathfrak{e}$, equal to 0 q.e. on $\mathfrak{e}$ and $\log(|z|) + O(1)$ at ∞.

Ratio Asymptotics

Szegő's Asymptotic Theorem for OPUC says $\Phi_n^*(z) \to D(0)D(z)^{-1}$ as $n \to \infty$ so $\Phi_{n+1}^*/\Phi_n^* \to 1$. We state without proof

Krushchev's Theorem (see [OPUC2], Section 9.5).

$\Phi_{n+1}^*(z)/\Phi_n^*(z)$ *converges uniformly on each* $\{z \mid |z| < 1 - \varepsilon\}$ *if and only if either*

For $\ell = 1, 2, \dots$, $\lim_{n\to\infty} \alpha_{n+\ell}\, \alpha_n = 0$; *limit is then 1.*

OR $\exists\, a \in (0,1]$ *and* $\lambda \in \partial\mathbb{D}$ *so* $\lim_{n\to\infty} |\alpha_n| = 0$, $\lim_{n\to\infty} \bar{\alpha}_{n+1}\, \alpha_n = a^2\, \lambda$

and then limit $\frac{1}{2}\left[(1+\lambda z) + \sqrt{(1-z\lambda)^2 + 4a^2\,\lambda z}\right]$.

Ratio Asymptotics

For OPRL, we have

Simon's Theorem (*J. Approx. Th.* **128** (2004), 198–217). For OPRL if $\lim_{n\to\infty} \frac{P_{n+1}(z)}{P_n(z)}$ exists at a single point in $\mathbb{C} \setminus R$, it exists at all points and this happens if and only if for some $a \in [0,\infty)$, $b \in \mathbb{R}$

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b$$

and the limit is

$$\frac{1}{2}\left[(z-b)+\sqrt{(z-b)^2-4a^2}\right] \quad \text{(root with } \sqrt{} = z \text{ near } \infty\text{)}$$

Ratio Asymptotics

Closely related to ratio asymptotics (because the conclusions imply ratio asymptotics) are

Rakhmanov's Theorem. *If* $d\mu = f(\theta)\frac{d\theta}{2\pi} + d\mu_s$ *and* $f(\theta) > 0$ *for a.e.* θ, *then* $\alpha_n \to 0$.

Denisov–Rakhamanov Theorem. *If* $d\mu = f(x)\,dx + d\mu_z$ *and* $f(x) > 0$ *on* $[-2,2)$ *and* $\sigma_{(\text{ess})}(J) = [-2,2]$, *then* $a_n \to 1$, $b_n \to 0$.

I hope to say more about this in Lecture 11 or 12.

Moral is ratio and Szegő asymptotics unusual. Expect oscillations.