Spectral Theory of Orthogonal Polynomials

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Lecture 2: Szegö Theorem for OPUC

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References

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[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

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Szegő's Theorem as a Variational Principle

Szegő's Theorem was proven by him in 1914 as a statement about Toeplitz Determinants as we discuss below.

In 1920–21, he rephrased it as a variational principle in OPUC. This (two-part) paper essentially invented the general theory of OPUC.

In these papers, Szegő assumed $d\mu$ was purely a.c. The addition of a singular continuous part is a discovery of Verblunsky in 1934–35 but his work was largely ignored and he didn't get credit until about fifteen years ago when, in a different context, Killip and Simon rediscovered his proof and then his paper.

Szegő's Theorem as a Variational Principle

Φ_n has a variational form. Since $\Phi_n =$ Proj of z^n onto the orthogonal complement of $\{1, \dots, z^{n-1}\}$,

$$\|\Phi_n\| = \text{dist of } z^n \text{ to span of } \{1, \dots, z^{n-1}\}$$

$$= \min\{\|P\| \mid P \text{ monic }, \deg P = n\}$$

$$= \min\{\|P^*\| \mid P(0) = 1, \deg P = n\}$$

since P monic $\Leftrightarrow P^*(0) = 1$.

This implies $\|\Phi_{n+1}\| \leq \|\Phi_n\|$ which is obvious from $\|\Phi_n\| = \rho_0 \, \rho_1 \dots \rho_{n-1}$ and $\rho_j \leq 1$.

Szegő's Theorem as a Variational Principle

Thus, clearly, $\lim_{n\to\infty}\|\Phi_n\|$ exists and

$$\lim_{n\to\infty}\|\Phi_n\| = \inf\{\|P\| \mid P(0)=1, P \text{ is a polynomial }\}$$

Szegő Theorem for OPUC. *Let*

$$d\mu = f(\theta)\,\frac{d\theta}{2\pi} + d\mu_s$$

be an arbitrary probability measure. Then

$$\inf\{\|P\|^2 \mid P(0)=1, P \text{ is a polynomial }\} = \exp\biggl(\int \log f(\theta)\frac{d\theta}{2\pi}\biggr)$$

Szegő's Theorem as a Variational Principle

This innocuous-looking theorem will have remarkable consequences as we'll see, in part because it has multiple equivalent forms.

Because $\int f(\theta)\frac{d\theta}{2\pi} < \infty$, the integral cannot diverge to $+\infty$, but it can to $-\infty$ in which case, we interpret $\exp(***)$ as 0. Indeed, by Jensen's inequality and the concavity of log, the integral is non-positive and the exponential in $[0,1]$ as it must be given that $\|\Phi_0\| = 1$.

One remarkable aspect of this theorem is that $d\mu_s$ doesn't enter!

Before turning to the proof, we consider some equivalent forms and consequences.

Szegő's Theorem as a Sum Rule

As we've seen, $\|\Phi_n\| = \rho_1 \dots \rho_{n-1}$ so

$$\lim \|\Phi_n\|^2 = \prod_{j=0}^{\infty} (1 - |\alpha_j|^2)$$

Szegő Theorem (Sum Rule Version). *If* $d\mu = f(\theta)\frac{d\theta}{2\pi} + d\mu_s$, *then*

$$\sum_{j=0}^{\infty} \log(1 - |\alpha_j|^2) = \int \log(f(\theta)) \frac{d\theta}{2\pi}$$

This is a precursor of KdV sum rules. It is clearly equivalent to the variational form.

Szegő's Theorem as a Sum Rule

Corollary. $\sum_{j=0}^{\infty}|\alpha_j|^2 < \infty \Leftrightarrow \int \log\bigl(f(\theta)\bigr)\,\frac{d\theta}{2\pi} > -\infty.$

A consequence of this is that $d\mu_s$ can be more or less arbitrary while one still has $\sum_{j=0}^{\infty}|\alpha_j|^2 < \infty$; for example, if $\int d\mu_s = \eta < 1$, $(1-\eta)\frac{d\theta}{2\pi} + d\mu_s = d\mu$ has $\sum_{j=0}^{\infty}|\alpha_j(\mu)| < \infty$.

This is remarkable because we'll see in a future lecture that $\sum_{j=0}^{\infty}|\alpha_j| < \infty \Rightarrow d\mu$ is purely a.c. and $\varepsilon < |f(\theta)| < \varepsilon^{-1}$ for some $\varepsilon > 0$ and all θ.

It is also remarkable because it is not easy to construct operators with mixed spectrum and potential decay.

Szegő's Theorem and Toeplitz Determinant Asymptotics

Given $\{c_n\}_{n=-\infty}^{\infty}$, the corresponding $N \times N$ Toeplitz matrix $T_N(c)$ has the form

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{-1} & c_0 & \cdots & c_N \\ \vdots & & \ddots & \vdots \\ c_{-N+1} & c_{-N+2} & \cdots & c_0 \end{pmatrix}$$

i.e., $\big(T_N(c)\big)_{ij} = c_{j-i}$. If μ is a measure, we set $c_j = \int e^{-ij\theta} d\mu(\theta)$ and write (μ is called the *symbol*)

$$D_N(\mu) = \det\big(T^{N+1}(\mu)\big)$$

Notice that in the $L^2(d\mu)$ inner product,

$$(T_N)_{kj} = \langle e^{ik\theta}, e^{ij\theta} \rangle = \langle z^k, z^j \rangle$$

Writing $\Phi_N = z^N +$ l.o. and using sums of rows and columns, one sees that

$$D_N(\mu) = \det\big(\langle \Phi_j, \Phi_k \rangle\big)_{0 \le j,\, k \le N}$$

$$= \|\Phi_0\|^2 \cdots \|\Phi_N\|^2$$

Since $\|\Phi_j\| \downarrow$, one sees that

$$\lim_{N\to\infty} D_N(\mu)^{1/N+1} = \lim_{N\to\infty} \|\Phi_N\|^2$$

Thus,

Toeplitz Determinant Form of Szegő's Theorem. *For any μ,*

$$\lim_{N\to\infty} \frac{1}{N+1} \log D_N(\mu) = \int \log f(\theta) \frac{d\theta}{2\pi}$$

Szegő's Theorem and Toeplitz Determinant Asymptotics

Aside: It is known that if $d\mu_s = 0$ and

$$\log\bigl(f(\theta)\bigr) \equiv \sum_{n=-\infty}^{\infty} \widehat{L}_n \, e^{in\theta}$$

and

$$\sum_{n=1}^{\infty} n|\widehat{L}_n|^2 < \infty$$

then

$$\log D_N(\mu) = (N+1)\widehat{L}_0 + \sum_{n=1}^{\infty} n|\widehat{L}_n|^2 + o(1)$$

This is the Strong Szegő Theorem. [OPUC1], Chap. 6 has many proofs of this.

When are Polynomials Dense in $L^2(\partial\mathbb{D}, d\mu)$?

By Weierstrass' Theorem, for any μ of compact support on $\mathbb{R}$, the polynomials in x are dense in $L^2(\mathbb{R}, d\mu)$.

But this is not true for $\partial\mathbb{D}$. Indeed, if $d\mu = \frac{d\theta}{2\pi}$, the closure of the polynomials are those functions in L^2 whose negative Fourier coefficient $\int e^{-in\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = 0$ for $n \leq -1$. On the other hand, we'll see soon that if $\mathrm{supp}(d\mu) \neq \partial\mathbb{D}$, the polynomials are dense.

When are Polynomials Dense in $L^2(\partial\mathbb{D}, d\mu)$?

Theorem (Kolmogorov-Krein). *If* $d\mu = f\,\frac{d\theta}{2\pi} + d\mu_s$, *then the polynomials in* z *are dense in* $L^2(\partial\mathbb{D}, d\mu)$ *if and only if* $\int \log f(e^{i\theta})\,\frac{d\theta}{2\pi} = -\infty$.

They found this because this density result was relevant to their theory of prediction for stochastic processes.

Given Szegő's Theorem, the proof is almost trivial for

$$\inf_P \|z^{-1} - P\|^2_{L^2} = \inf_P \|1 - zP\|^2_{L^2}$$

$$= \inf_{Q|Q(0)=1} \|Q\|^2_{L^2} = \exp\Big(\int \log f\,\frac{d\theta}{2\pi}\Big)$$

When are Polynomials Dense in $L^2(\partial\mathbb{D}, d\mu)$?

So $z^{-1} \in$ closure of polys $\Leftrightarrow \int \log f \frac{d\theta}{2\pi} = -\infty$.

Thus, if the integral is finite, $z^{-1} \notin$ closure of polys and thus, polynomials are not dense.

On the other hand, if $z^{-1} = \lim P_n$, then $z^{-2} = \lim_{n\to\infty} P_n \left[\lim_{m\uparrow\infty} P_m\right]$ so all polynomials in z and z^{-1} are in closure of polys and they are dense (by Weierstrass' other density theory).

Krein used this to show (see [SzThm], p. 319) that on $\mathbb{R}$, if $d\rho = F\,dx + d\rho_\nu$, then $\{e^{i\alpha x}\}_{\alpha\geq 0}$ are dense in $L^2 \Leftrightarrow \int \frac{\log F(x)}{1+x^2}\,dx = -\infty$. This, in turn, implies that if $\int |x|^n\,d\rho(x) < \infty$, the moment problem is indeterminate if the integral is finite, for example,

$$d\rho(x) = e^{-|x|^\alpha}\,dx, \quad \alpha < 1$$

Strategy of the Proof

As with all good proofs of equalities, we'll prove two inequalities. We'll use "entropy term" for $\exp\left[\int \log f \frac{d\theta}{2\pi}\right]$ for reasons that will become clear soon.

The proof that $\lim_{n\to\infty} \|\Phi_n^*\|$ is an upper bound will be variational. We'll show that for any polynomial with $P(0) = 1$, we have $\|P\| \geq$ entropy term.

Strategy of the Proof

The lower bound on the entropy term will come from the fact that $\mu \mapsto$ entropy term is weakly upper-semicontinuous (usc), i.e., $\mu_n \to \mu \Rightarrow S(\mu) \geq \limsup S(\mu_n)$.

We'll prove that $S(\mu) = \prod_{j=0}^{\infty}(1 - |\alpha_j|^2)^{1/2}$ for Bernstein–Szego measures by direct calculation and then use this and usc to get the other inequality.

Upper Bound

Lemma. For any polynomial P, with $P(0) \neq 0$, we have that

$$\int \log|P(e^{i\theta})| \frac{d\theta}{2\pi} \geq \log|P(0)|$$

Remark. One proof notes that $\log\big(P(z)\big)$ is subharmonic.

Proof. If $\{z_j\}_{j=1}^m$ are zeros in $\mathbb{D}$, let

$$Q(z) = \prod_{j=1}^m \frac{1 - \bar{z}_j z}{z - z_j} P(z)$$

Then $\log Q(z)$ is analytic in $\mathbb{D}$, so

Upper Bound

$$\log|Q(0)| = \lim_{r\uparrow 1} \int \log|Q(re^{i\theta})| \, \frac{d\theta}{2\pi} = \int \log|Q(e^{i\theta})| \, \frac{d\theta}{2\pi}$$

$$= \int \log|P(e^{i\theta})| \frac{d\theta}{2\pi}$$

But, $|Q(0)| = \prod_{j=1}^{m} |z_j|^{-1} \, |P(0)| \geq |P(0)|$.

Upper Bound

For any polynomial, P, with $P(0) \neq 0$, $d\mu = f\,\frac{d\theta}{2\pi} + d\mu_s$, we have

$$\int |P(e^{i\theta})|^2\, d\mu(\theta) \geq \int |P(e^{i\theta})|^2 f(\theta)\,\frac{d\theta}{2\pi}$$

$$= \int \exp\bigl[2\log|P(e^{i\theta})| + \log\bigl(f(\theta)\bigr)\bigr]\,\frac{d\theta}{2\pi}$$

$$\geq \exp\biggl(\int 2\log\bigl(|P(e^{i\theta})|\frac{d\theta}{2\pi}\bigr)\exp\bigl(\int \log f\,\frac{d\theta}{2\pi}\bigr)\biggr)$$

(by Jensen)

$$\geq |P(0)|^2 \exp\biggl(\int \log|f(\theta)|\,\frac{d\theta}{2\pi}\biggr)$$

by the Lemma. Thus

$$\inf_{P|P(0)=1} \int |P(e^{i\theta})|^2\, d\mu \geq \exp\bigl(\int \log\bigl(f(\theta)\bigr)\bigr)\,\frac{d\theta}{2\pi}$$

Upper Bound

One can also get a variational upper bound to complete the proof. The idea is to consider the function

$$D(z) = \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\log(f(\theta))\frac{d\theta}{4\pi}\right)$$

Formally, and we'll see later that D is actually in $H^2(\mathbb{D})$ and has boundary values, $D(e^{i\theta}) = \lim_{r\to\infty} D(re^{i\theta})$ exists for a.e. θ and $|D(e^{i\theta})|^2 = f(\theta)$.

If $d\mu_s = 0$, we have $P(z) = D(0)/D(z)$ has $P(0) = 0$ and

$$\int |P(z)|^2\, d\mu = D(0)^2 \int f(\theta)^{-2}\left[f(\theta)\frac{d\theta}{2\pi}\right] = D(0)^2$$

$$= \exp\left(\int \log(f(0))\frac{d\theta}{2\pi}\right)$$

Upper Bound

P isn't a polynomial but one can approximate by polynomials . Handling $d\mu_s$ is a separate issue, but it can be done (see [OPUC1], Section 2.5 and [SzThm], Section 2.12).

The Bernstein–Szegő Case

Suppose $\alpha_j = 0$ for $j \geq N$. Then, we've seen that

$$d\mu = f(\theta)\frac{d\theta}{2\pi}, \quad f(\theta) = |\varphi_N^*(e^{i\theta})|^{-2}$$

Thus,

$$\log f(\theta) = -2\log|\varphi_N^*(e^{i\theta})| = \log\|\Phi_N^*\|^2 - 2\log|\Phi_N^*(e^{i\theta})|$$

Since $\Phi_N^*(z)$ is analytic in a nbhd of $\bar{\mathbb{D}}$, so is $\log\big(\Phi_N^*(z)\big)$, so

$$\int \frac{d\theta}{2\pi}\log|\Phi_N^*(e^{i\theta})| = \log|\Phi_N^*(0)| = 0$$

Thus,

$$\int \log f(\theta)\frac{d\theta}{2\pi} = \log\|\Phi_N^*\|^2 = \log\prod_{j=0}^{N-1}\big(1-|\alpha_j|^2\big)^{1/2}$$

proving Szegő's Theorem in this case.

The Szegő Integral as an Entropy

Given two prob. measures on $\partial\mathbb{D}$, we define their relative entropy by

$$S(\mu \mid \nu) = \begin{cases} -\infty & \text{if } \mu \text{ is not } \nu\text{-a.e.} \\ -\int \log\left(\frac{d\mu}{d\nu}\right) d\mu & \text{if } \mu \text{ is } \nu\text{-a.e.} \end{cases}$$

For example, $S(g d\nu \mid d\nu) = -\int g \log(g) d\nu$

Usually ν is fixed and we vary μ.

The Szegő Integral as an Entropy

We claim that

$$S\left(\frac{d\theta}{2\pi}\,\middle|\, f\frac{d\theta}{2\pi} + d\mu_s\right) = \int \log\bigl(f(\theta)\bigr)\frac{d\theta}{2\pi}$$

For μ is ν-a.e. iff $f(\theta) \neq 0$ for $\frac{d\theta}{2\pi}$-a.e. θ. If $f(\theta) = 0$ on a positive Lebesgue measure set, the integral is $-\infty$, so both sides are $-\infty$.

If $f(\theta) \neq 0$ for a.e. θ, $\frac{d\mu}{d\nu} = f^{-1}\chi_S$ where χ_S is a set with $d\mu_s(S) = 0$ and $|S| = 1$. Clearly

$$-\int \log\left(\frac{d\mu}{d\nu}\right) = \int \log\bigl(f(\theta)\bigr)\frac{d\theta}{2\pi}$$

Variational Principle for S

Here is a basic fact which we'll make plausible but not formally prove (but see Section 2.2 of [SzThm]).

Theorem. Let $\mathcal{E}(\partial\mathbb{D})$ be the continuous strictly positive functions on $\partial\mathbb{D}$. Then

$$S(\mu \mid \nu) = \inf_{f\in\mathcal{E}(\partial\mathbb{D})} \mathcal{S}(f;\mu,\nu)$$

where

$$\mathcal{S}(f;\mu,\nu) = \int f(x)d\nu(x) - \int 1 + \log\bigl(f(x)\bigr)\, d\mu$$

Proof. If $d\mu = gd\nu$ with $g \in \mathcal{E}(\partial\mathbb{D})$, then

$$\mathcal{S}(g;gd\nu,\nu) = 1 - 1 - \int \log\bigl(g(x)\bigr)\, d\mu = S(gd\nu \mid \nu)$$

By an approximation argument (and control of $d\mu_s$) one obtains

$$S(\mu \mid \nu) \geq \inf \mathcal{S}$$

Variational Principle for S

Let's prove $\mathcal{S}(f;\mu,\nu) \geq S(\mu \mid \nu)$ in case $d\mu_s = 0$ so

$$d\nu = g^{-1} d\mu$$

so that

$$\mathcal{S}(f;\mu,\nu) = \int Q_{g(x)}\big(f(x)\big)\, d\mu(x)$$

where

$$Q_b(x) = xb^{-1} - 1 - \log x$$

Then

$$Q_b'(x) = b^{-1} - x^{-1}, \quad Q_b''(x) = x^{-2} \geq 0$$

so Q_b is convex, $Q_b'(b) = 0$, so $Q_b(x) \geq Q_b(b)$, i.e.,

$$Q_b(x) \geq -\log(b)$$

Thus

$$\mathcal{S}(f;\mu,\nu) \geq -\int \log\big(g(x)\big)\, d\mu(x) = S(\mu \mid \nu)$$

Variational Principle for S

For each fixed f in $\mathcal{E}(\partial\mathbb{D})$, $\mathcal{S}(f;\mu,\nu)$ is linear and weakly continuous so the inf is concave and weakly usc, i.e.

Theorem. *$S(\mu \mid \nu)$ is jointly converse and jointly weakly usc in μ and ν.*

Corollary. *Define $Sz(\mu) = \int \log f \frac{d\theta}{2\pi}$ if $d\mu = f \frac{d\theta}{2\pi} + d\mu_s$. Then $\mu \mapsto Sz(\mu)$ is weakly usc.*

The end of the proof

Let μ have Verblunsky coefficients, $\{\alpha_n\}_{n=0}^{\infty}$. Let μ_n be the Bernstein–Szegő approximation.

We've proven above that

$$Sz(\mu_n) = \prod_{j=0}^{n-1} \rho_j^2$$

By weak usc

$$Sz(\mu) \geq \overline{\lim}\, Sz(\mu_n) = \prod_{j=0}^{\infty} \rho_j^2$$

which is the other inequality that we needed to prove.