Spectral Theory of Orthogonal Polynomials

Barry Simon
IBM Professor of Mathematics and Theoretical Physics
California Institute of Technology
Pasadena, CA, U.S.A.

Lectures 11 & 12: Selected Additional Topics, I, II

- Lecture 9: Fuchsian Groups and Finite Gaps, I
- Lecture 10: Fuchsian Groups and Finite Gaps, II
- Lecture 11: Selected Additional Topics, I
- Lecture 11: Selected Additional Topics, II

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Topics

- Right Limits and Essential Spectrum, [SzThm], §7.3
- Reflectionless Jacobi Matrices and Remling's Theorem, [SzThm], §7.4–7.6
- The Magic Formula, [SzThm], Chap. 8
- CMV Matrix, [OPUC1], Chap. 4
- Clock Behavior and Universality, [SzThm], §2.16, 2.17, 3.11, 3.12, 5.11
- Toda Flows, [SzThm], Chap. 6
- Regular Measures, [SzThm], §5.9
- Ratio Asymptotics and Weak Limits, [OPUC2], §9.5–9.8

Right Limits

Right limits were introduced and studied by Last–Simon in two papers [Inv. Math. **135** (1999), 329–367; J. Anal. Math. **98** (2006), 183–220].

Given a set of bounded (one-sided) Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$, we say a two-sided sequence $\{\tilde{a}_n, \tilde{b}_n\}_{n=-\infty}^{\infty}$ is a right limit if there exists $m_j \to \infty$ so that for all ℓ, as $j \to \infty$,

$$a_{m_j+\ell} \to \tilde{a}_\ell; \quad b_{m_j+l} \to \tilde{b}_\ell$$

The set of all right limits is denoted $\mathcal{R}(J)$. By compactness, it is non-empty and a closed subset of $[-R, R]^\infty$ in the product topology ($R = \sup_n |a_n| + |b_n|$).

Right Limits

We say $\lambda \in \sigma_{\infty,pp}(\widetilde{J})$ if and only if $\exists\, u \in L^\infty$, $u \not\equiv 0$, with $\widetilde{J}u = \lambda u$. By a Weyl sequence argument, $\sigma_{\infty,pp}(\widetilde{J}) \subset \sigma(\widetilde{J})$.

Last–Simon proved ($\sigma_{\infty,pp}(J)$ is from [SzThm] book)

Theorem. $\sigma_{\text{ess}}(J) = \cup_{J_r \in \mathcal{R}(J)} \sigma(J_r) = \cup_{J_r \in \mathcal{R}(J)} \sigma_{\infty,pp}(J_r)$

That $\sigma(J_r) \subset \sigma_{\text{ess}}(J)$ is easy. If u_n has finite support with $\|(J_r - \lambda)u\| < \varepsilon$, then with $u_n^{(j)} = u_{m_j+n}$, we have $\overline{\lim}_{j\to\infty} \|(J-\lambda)u^{(j)}\| \leq \varepsilon$ and $u^{(j)} \to 0$ weakly.

The other parts are not hard but somewhat involved.

Last–Simon also proved $\Sigma_{\text{ac}}(J) \subset \cap_{J_r} \Sigma_{\text{ac}}(J_r)$.

Reflectionless Jacobi Matrices

A reflectionless Jacobi matrix on $\mathfrak{e} \subset \mathbb{R}$ is a whole line matrix obeying

$\forall\, n,\ G_{nn}(x + i0) \in i\mathbb{R}$ for a.e. $x \in \mathfrak{e}$.

It is sufficient that this holds for three successive n's.

By the reflection principle (different reflection!), we have that if $(a, b) \subset \mathfrak{e}$, then G_{nn} is continuous on (a, b), so J has purely a.c. spectrum on (a, b).

Reflectionless Jacobi Matrices

Given J, a whole line Jacobi matrix, J_+ and J_- are the Jacobi matrices with parameters $\{a_n, b_n\}_{n=1}^{\infty}$ and $\{a_{-n}, b_{-n+1}\}_{n=1}^{\infty}$ and $\widetilde{J}$ with parameters $\{a_{-n-1}, b_{-n}\}_{n=1}^{\infty}$.

$J_{+/-}$ are the two pieces we get by setting $a_0 = 0$ and J_+ and $\widetilde{J}$ by setting $a_0 = a_{-1} = 0 = b_0$. $m^{\pm}, \tilde{m}$ are their m-functions. The $u_{\pm}$ formulae for G and m yield

$$G_{00}(z) = -\left(a_0^2 m^+(z) - m^-(z)^{-1}\right)^{-1}$$

$$G_{00}(z) = -\left(z - b_0 + a_{-1}^2 \tilde{m} + a_0^2 m^+\right)^{-1}$$

A basic fact is

Theorem. *J is reflectionless if and only if for a.e. $x \in \mathfrak{e}$*

$$\overline{m^+(x+i0)} = \big(a_0^2 m^-(x+i0)\big)^{-1}$$

$$\iff \overline{a_0^2 m^+(x+i0)} = b_0 - x - a_1^2 \tilde{m}(x+i0)$$

These conditions imply $G_{00}(x+i0)$ is pure imaginary since $\mathrm{Re}\big(a_0^2 m^+(x+i0) - (m^-(x+i0))^{-1}\big) = 0$.

Reflectionless Jacobi Matrices

To get full result, one shows that Weyl solutions for J^+ and $\widetilde{J}$, call them u_n^+ and u_n^-, have a.c. boundary values (since m does). $\operatorname{Re} G_{nn}(x+i0)$ for $n = 0, \pm 1$ is equivalent to (all at $x + i0$) $\operatorname{Im}(u_n^+ u_n^-) = 0$ for $n = \pm 1$ and W (= Wronskian) has $\operatorname{Re} W = 0$ and that this is equivalent to $u_n^- = \overline{u_n^+}$.

Reflectionless Isospectral Torus

Suppose now $\mathfrak{e} = \cup_{j=1}^{\ell+1}[\alpha_j, \beta_j]$ is a finite gap set. Let J be reflectionless on $\mathfrak{e}$ with $\sigma(J) = \mathfrak{e}$. Let $m^+(z)$ be the m-function for the half-line operator J^+.

Since m^+ is real in the gaps and on $(-\infty, \alpha_1)$ and $(\beta_{\ell+1}, \infty)$, we have that

$$m^+(x+i0) = \overline{m^+(x-i0)} = [a_0^2 m^-(x-i0)]^{-1}$$

This implies that m^+ defined on $\mathcal{S}_+$ can be analytically continued to $\mathcal{S}_-$ by defining it to be $(a_0^2 m^-(z))^{-1}$ on $\mathcal{S}$.

Since m^- has a zero at ∞, $(m^-)^{-1}$ has a pole there.

Reflectionless Jacobi Matrices

The only other possible poles are in the gaps.

$(a_0 m^-(z))^{-1} = b_0 - z - a_1^2 \tilde{m}(z)$, so $(a_0^2 m^-)^{-1}$ has a pole $\Leftrightarrow \tilde{m}$ does.

$G_{00} = -(z - b_0 + a_{-1}^2 \tilde{m} + a_0^2 m^+)^{-1}$ shows that if $\tilde{m}$ or m^+ have poles at $x_0 \in (\beta_j, \alpha_{j+1})$, then G_{00} has a zero there.

Since G_{00} is monotone (and bounded) on (β_j, α_{j+1}), it has at most one zero there. If it has a zero at x, either m^+ or $\tilde{m}$ or both have a pole there.

Reflectionless Jacobi Matrices

If m^+ and $\tilde{m}$ both have poles there, one can show the Weyl solutions agree and vanish at $n = 0$ which implies the whole line problem has an eigenvalue at x, contrary to our hypothesis that $\sigma(J) = \mathfrak{e}$.

Thus, at a zero of G_{00} in a gap m^+ has a pole at precisely one of the points $z_\pm$ with $\pi(z_\pm) = x_0$. A further analysis shows that if G_{00} has no zeros in a given gap, it vanishes at one end or the other where one has a simple pole in the local coordinates.

Reflectionless Jacobi Matrices

We thus see if J is reflectionless on $\mathfrak{e}$ and $\sigma(J) = \mathfrak{e}$, m^+ is a minimal Herglotz function. Conversely, if m^+ is a minimal Herglotz function, $(a_0^2 m^+(\tau(z)))^{-1}$ defines a J^- (where a_0^2 is picked so $(a_0^2 m^+(\tau(z)))^{-1} \sim -z^{-1}$ near ∞_+) and J^+, a_0, J^- fit together into a reflectionless J. We have thus proven:

Theorem. *There is a 1-1 correspondence between the isospectral torus on $\mathfrak{e}$ (defined as minimal Herglotz functions) and reflectionless J's.*

Remling's Theorem

Remling proved the following remarkable theorem:

Remlings Theorem. *If J is a half-line Jacobi matrix and $\mathfrak{e} \equiv \Sigma_{\mathrm{ac}}(J) \neq \emptyset$, then any right limit, J_r, is reflectionless on $\mathfrak{e}$.*

There is an attractive intuition: J_r repeats infinitely often far from each other. If there were any reflection, a wave would get trapped by the infinitely many reflections but by Riemann–Lebesgue, a.c. wave packets get "out" to ∞.

Remling's Theorem

Alas, no one has a proof using this intuition. The only proof we have is Remling's original proof which takes 20 dense pages in [SzThm] and which I can follow but don't understand!

Denisov–Rakhmanov–Remling Theorem

Theorem (Rakhmanov [Math. Sb. **32** (1977), 199–213; **46** (1983), 105–117]). *Let* $d\mu = f\frac{d\theta}{2\pi} + d\mu_s$ *on* $\partial\mathbb{D}$ *with* $f(\theta) > 0$ *for Lebesgue a.e.* θ. *Then* $\alpha_n(d\mu) \to 0$.

Theorem (Denisov [Proc. A.M.S. **130** (2004), 847–852). *Let* $d\mu = f\,dx + d\mu_s$; $\operatorname{ess\,supp}(d\mu) = [-2,2]$; $f(x) > 0$ *for a.e.* $x \in [-2,2]$.

Then $a_n(d\mu) \to 1$, $b_n(d\mu) \to 0$.

Theorem (Damanik–Killip–Simon, Ann. Math. **171** (2010), 1931–2010). *Let* $\mathfrak{e}$ *be a finite gap set with each band having rational harmonic measure. Let* $d\mu = f\,dx + d\mu_s$; $\operatorname{ess}(d\mu) = \mathfrak{e}$; $f(x) > 0$ *for a.e.* $x \in \mathfrak{e}$. *Then, as* $\ell \to \infty$, $\{a_{n-\ell}, b_{n-\ell}\}_{n=1}^{\infty}$ *approaches the two-sided (periodic) isospectral torus for* $\mathfrak{e}$.

Denisov–Rakhmanov–Remling Theorem

Theorem (Remling [Ann. Math **174** (2011), 125–171]). *The last theorem holds for any finite gap set, $\mathfrak{e}$.*

Remarks. 1. This was conjectured by Damanik–Killip–Simon.

2. This implies the Denisov result and also Rakhmanov once one has an extension of Remlin to OPUC which was accomplished by Breuer, Ryckman, Zinchenko [Comm. Math. Phys. **292** (2009), 1–28].

The DRR Theorem is an immediate consequence of Remling's Theorem and the analysis of J's reflectionless on $\mathfrak{e}$ with $\sigma(J) = \mathfrak{e}$.

The Magic Formula

Theorem (Damanik–Killip–Simon [Ann. Math. **171** (2010), 1931–2010]). *Let Δ be the discriminant associated to a finite gap set $\mathfrak{e} \subset \mathbb{R}$ with rational harmonic measures (Δ is degree p, measures are q_j/p with no common factor for $p \cup \{q_j\}_{j=1}^{\ell+1}$). Let J be a whole line Jacobi matrix and $S : \ell_2 \to \ell_2$ by $(Su)_n = u_{n-1}$. Then*

$$\Delta(J) = S^p + S^{-p} \Leftrightarrow J \in \text{ isospectral torus for } \mathfrak{e}$$

DKS called this the magic formula.

The Magic Formula

Recall for $J \in$ isospectral torus, $\Delta(E(\theta)) = 2\cos\theta$ where $E(\theta)$ are eigenvalues of Floquet solutions with $u_{n+p} = e^{i\theta} u_n$. In a suitable spectral representation S^p is multiplication by $e^{i\theta}$ and J by $E(\theta)$ so that the above shows that $\Delta(J) = S^p + S^{-p}$.

The Magic Formula

For the converse, if $\Delta_{\mathfrak{e}}(J) = S^p + S^{-p}$, then $[J, \Delta_{\mathfrak{e}}(J)]$ so $[J, S^p + S^{-p}]$. Using that J has finite width, this implies (an argument of Naiman [Soviet Math. Dokl. **3** (1962), 383–385]) $[J, S^p] = 0$, i.e., J has periodic p.

Thus, $\Delta_J(J) - \Delta_{\mathfrak{e}}(J) = 0$. Again, using finite width of J, $p(J) = 0$ for a polynomial $\Rightarrow p = 0$. Thus $\Delta_J = \Delta_{\mathfrak{e}} \Rightarrow J \in$ isospectral torus.

The Magic Formula

To use this idea, note that in general, if J is a Jacobi matrix, $\Delta(J)$ is not tri-diagonal but $2p+1$ diagonal (since $(J^\ell)_{k_j} \neq 0 \Rightarrow |k-j| \leq \ell$) which can be thought of as $p \times p$ block tridiagonal.

The DKS strategy is to extend results on approach to $[-2, 2]$ to the matrix-valued case.

The Magic Formula

For example, to get the extension of Denisov, if J has $\Sigma_{\rm ac}(J) = \mathfrak{e}$, then $\Delta(J)$ has multiplicity p a.c. spectrum on $[-2,2]$. By a matrix extension of Denisov, this implies all right limits of $\Delta(J)$ are $S^p + S^{-p}$. By the Magic Formula, this proves any right limit J_r obeys $\Delta(J_r) = S^p + S^{-p}$, so $J_r \in$ isospectral torus.

The Magic Formula

DKS are thus able to prove

- DRR for periodic (also follows from Remling)
- Shohat-Nevai for perturbations of isospectral torus (also follows from CSZ)
- Lieb–Thirring and so Nevai conjecture for perturbations of periodic (also follows from Frank–Simon)
- Killip–Simon for perturbations of periodic with all gaps open (only known proof!)

CMV Matrix

In 2003, Cantero, Moral, and Velázquez [Linear Alg. Appl. **362** (2003), 29–51] found the "right" matrix representation for OPUC 82 years after Szegő invented OPUC! Its usefullness for spectral theory of OPUC was found by Golinskii–Simon and presented in Section 4.3 of [OPUC].

Just as Jacobi matrices come from general self-adjoint operators with cyclic vector, CMV matrices come from unitary matrices with cyclic vector and in this form (without the OPUC connection) they were found in the numerical linear algebra community in 1991 by Bunse-Gerstner–Elsner [Linear Alg. Appl. **154** (1991), 741–678].

CMV Matrix

For OPUC, orthonormalized z^n may not give a basis for $L^2(\partial\mathbb{D}, d\mu)$. CMV had the idea of orthonormalizing, $1, z, z^{-1}, z^{-2}, \dots$ which always gives a basis!

Remarkably, the resulting basis, $\{\chi_j\}_{j-0}^{\infty}$ can be expressed in terms of the φ_n's and φ_n^*'s. If

$$\sigma_n = \chi_{2n}, \quad \tau_n = \chi_{2n-1}; \quad n = 0, 1, 2, \dots (n \geq 1 \text{ for } \ =)$$

then $\sigma_n = z^{-n}\varphi_{2n}^*, \quad \tau_n = z^{-n+1}\varphi_{2n-1}$

CMV Matrix

If instead, one orthonormalizes, $1, z, z^{-1}, z^{-2}, \ldots$, one gets an ON basis, $\{x_j\}_{j=0}^{\infty}$ and if $s_n = x_{2n}$, $n = 0, 1, 2, \ldots$, $t_n = x_{2n-1}$, $n = 1, 2, \ldots$, then

$$s_n = z^{-n}\varphi_{2n}, \quad t_n = z^{-n}\varphi_{2n-1}^{*}$$

One defines the CMV and alternate CMV matrices by

$$C_{ij}(d\mu) = \langle \chi_i, z\chi_j \rangle; \quad \widetilde{C}_{ij}(d\mu) = \langle x_i, zx_j \rangle$$

CMV Matrix

One defines $\mathcal{L}_{ij}(d\mu) = \langle x_i, \chi_j \rangle$, $m_{ij}(d\mu) = \langle \chi_i, z x_j \rangle$ so one has the LM factorization

$$\mathcal{C} = \mathcal{L}\mathcal{M}, \quad \widetilde{\mathcal{C}} = \mathcal{M}\mathcal{L}$$

If $\Theta_j = \left(\begin{smallmatrix} \bar{\alpha}_j & \rho_j \\ \rho_i & -\alpha_j \end{smallmatrix} \right)$

then $\mathcal{L} = \Theta_0 \oplus \Theta_2 \oplus \Theta_4 \oplus \dots$

and $\mathcal{M} = \mathbb{1}_{1\times 1} \oplus \Theta_1 \oplus \Theta_3 \oplus \dots$

CMV Matrix

Thus, $\mathcal{C}$ is 5-diagonal with structures of 4×2 blocks.

Matrix elements are quadratic in α's and ρ's, explicitly

$$\mathcal{C} = \begin{pmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

CMV Matrix

Consequences of CMV

- New proof of Verblunsky's Theorem
- Trace class theory:

$$\sum_{j=0}^{\infty}|\alpha_j-\tilde{\alpha}_j| < \infty \Rightarrow \mathcal{C}(\alpha)-\mathcal{C}(\tilde{\alpha}) \in \ell_1 \Rightarrow \Sigma_{\text{ac}}(\alpha) = \Sigma_{\text{ac}}(\tilde{\alpha})$$

- Weyl Theorem:

$$\alpha_j - \beta_j \to 0 \Rightarrow \sigma_{\text{ess}}(\alpha) = \sigma_{\text{ess}}(\beta)$$

- New proof of Geronimus Relatins [Killip–Nenciu, IMRN **50** (2004), 2665-2701]
- Haar on CUE induced measures on Verblunsky coefficients [Killip–Nenciu, IMRN **50** (2004), 2665-2701]

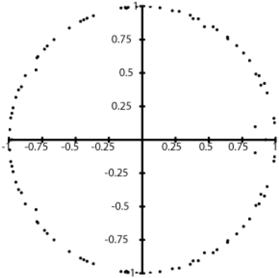
Fine Structure of OP Zeros

We've discussed the DOS but that only tells part of the story of the distribution of zeros for OPs.

In the next two slides, I'll show you the zeros for two different sets of Verblunsky coefficients. One has the α's iidrv—unfortunately uniform in a real interval (hence the complex conjugate symmetry) rather than uniform on some circle. I'm not sure of the DOS.

The other is $\alpha_n = (\frac{3}{4})^{n+1}$ where the DOS is known (a result of Mhaskar-Saff) to be uniform on the circle of radius $3/4$.

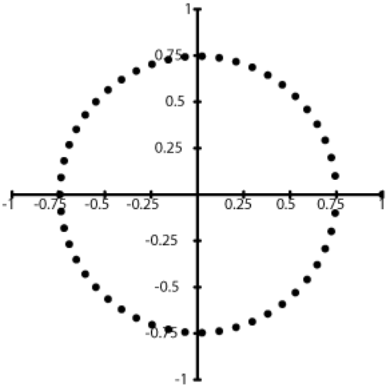
Here is the random case—which looks irregular:



In the rotation symmetry case, it is known to be Poisson [Stoiciu, JAT **139** (2006) 29–64; Davies–Simon, JAT **141** (2006), 189–213]

Fine Structure of OP Zeros

And here is the exponential decay case–which looks regular:



In this case, the zeros are asymptotically equally spaced—which I called clock spacing.

Fine Structure of OP Zeros

I want to focus on the clock-spacing case, but for OPRL, not OPUC.

When $a_n \to 1$, $b_n \to 0$, the DOS will be the equilibrium measure, $\rho_{\mathfrak{e}}(x)dx$ for $\mathfrak{e} = [-2, 2]$, i.e.,

$$\rho_{\mathfrak{e}}(x) = \pi^{-1}(4 - x^2)^{-1/2}$$

so clock only means locally equally spaced.

If $x_n^{(j)}(E_0)$ is the zeros of $p_n(x)$ with $x_n^{(-2)} < x_n^{(-1)} < x_n^{(0)} < E_0 < x_n^{(1)} \dots$ then clock spacing is

$$n\big[x_n^{(j+1)} - x_n^{(j)}\big] \to 1/\rho(E_0)$$

Fine Structure of OP Zeros

The earliest general clock-spacing results are due to Erdős–Turan [Ann. Math. **44** (1940), 510–553] and very general results are in Last–Simon [Comm. Pure Appl. Math. **61** (2008), 486–538].

But I want to focus on a wonderful approach of Lubinsky—actually two approaches, both based on the CD kernel [Lubinsky, Ann. Math. **170** (2009), 915–939; J. Anal. Math. **106** (2008), 373–394]

Fine Structure of OP Zeros

With his first method, Simon [J. Math. Anal. **105** (2008), 345–362] and Totik [Ark. Mat. **47** (2009), 361–391] obtained clock space for fairly general a.c. $d\mu$ on sets $\mathfrak{e}$ with $\overline{\mathfrak{e}^{\rm int}} = \mathfrak{e}$ and Avila–Last–Simon [Anal. PDE **3** (2010), 81–108], using Lubinsky's second method, even have some results for a.c. spectrum on positive measure Cantor sets.

Here, we'll focus on the underlying method.

The CD Kernel and Universality

A critical tool is the CD (for Christoffel Darboux) kernel

$$K_n(x,y) = \sum_{j=0}^{n} p_n(x)p_n(y)$$

which is the integral kernel in $L^2(\mathbb{R}, d\mu)$ of the projection into polynomials of degree n or smaller.

Theorem (CD Formula). *For* $x \neq y$

$$K_n(x,y) = \frac{a_{n+1}\big(p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)\big)}{x-y}$$

The CD Kernel and Universality

Proof. If $L_n(x,y) = a_{n+1}\bigl(p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)\bigr)$ recursion relation for $xp_n(x)$ times $p_n(y)$ minus recursion relation for $yp_n(y)$ times $p_n(x)$ says

$$(x-y)p_n(x)p_n(y) = L_n(x,y) - L_{n-1}(x,y)$$

This plus induction says that CP formula holds.

The CD Kernel and Universality

Universality on $[-2,2] = \mathfrak{e}$ says, for any compact interval $I \subset [-2,2]$,

$$\frac{1}{n+1} K_n(x_n, x_n) \to \frac{\rho_{\mathfrak{e}}(x_\infty)}{w(x_\infty)} \text{ if say } n|x - x_\infty| \leq A$$

uniformly in $x_\infty \in I$ and x_n's with A fixed.

$$\frac{K_n(x_\infty + \frac{a}{n}, x_\infty + \frac{b}{n})}{K_n(x_\infty, x_\infty)} \to \frac{\sin\bigl(\pi \rho_{\mathfrak{e}}(x_\infty)(b-a)\bigr)}{\pi \rho_{\mathfrak{e}}(x_\infty)(b-a)}$$

uniformly in $x_\infty \in I$, $|a| \leq A$, $|b| = A$.

The CD Kernel and Universality

Theorem (Freud–Levin Theorem). *Universality* $\Rightarrow$ *clock spacing.*

Proof. By CD formula, if $p_n(x) = 0$, $y \neq x$, then $p_n(y) = 0 \Leftrightarrow K_n(x, y) = 0$ (since $p_n(x) = 0 \Rightarrow p_{n+1}(x) \neq 0$).

Universality controls zeros of K_n for n large. Explicity $K_n(x, y) \neq 0$ if $|x - y| \leq \frac{\alpha}{\rho_{\mathfrak{e}}(x)}$ with $0 \leq \alpha < 1$ and n large.

Zeros of sin says at least one zero near zero $+\frac{k}{\rho_{\mathfrak{e}}(x_\infty)}$ and above says no more than 1.

The CD Kernel and Universality

Lubinsky provided two methods of proving universality.

See his papers or Sections 3.11 and 3.12 of [SzThm].